A Quick Introduction to Complex Tori

Parth Shimpi

§ 0.1 About. This is a short summary of a lecture course on Abelian varieties by Tony Scholl. Complete notes for the said course can be found at `https://zb260.user.srcf.net/notes/III/abel.pdf`. These notes were written as a supplement to the notes from a lecture course on Hodge theory (`https://pas201.user.srcf.net/documents/2023-ukag-hodge-theory.pdf`), and should be read as such.

§ 1 Complex tori and their cohomology

It is a theorem that all commutative compact connected $\mathbb{R}$-Lie groups arise as real tori, i.e. quotients V/Λ where V is a real vector space and $\Lambda \subset V$ is a lattice (free $\mathbb{Z}$-module generated by a basis of V). If in addition we fix an isomorphism $V \cong \mathbb{C}^n$, then the quotient V/Λ is naturally a $\mathbb{C}$-manifold and the group operation is holomorphic. Complex Lie groups arising in this way are called *complex tori.*

Proposition 1.1. *Any compact connected $\mathbb{C}$-Lie group is a complex torus. In particular it is commutative.*

If $T = V/\Lambda$ is a complex torus, the corresponding lattice Λ is called the *period* of T. Note that V is the universal cover of the complex torus $T = V$, and since the fundamental group of a torus is Abelian we have natural isomorphisms $\pi_1(T, 0) \cong H_1(T, \mathbb{Z}) \cong \Lambda$.

Remark 1.2. It is not hard to show that two tori V/Λ and V'/Λ' are isomorphic if and only if there is a $\mathbb{C}$-linear map $V \to V'$ that restricts to an isomorphism $\Lambda \to \Lambda'$. Thus every complex torus has form $\mathbb{C}^d/(\mathbb{Z} \oplus M\mathbb{Z})$ for some $d \times d$ complex matrix M (the *period matrix*) such that $\mathrm{Im} M$ has full rank. For instance, every elliptic curve is given by $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ for $\tau \in \mathbb{C} \setminus \mathbb{R}$.

§ 1.1 Hodge theory of tori. Let $T = V/\Lambda$ be a d-dimensional complex torus. Since $T \cong (S^1)^{2d}$, the Kunneth formula gives an isomorphism $H^n(T, \mathbb{Z}) \cong \bigwedge^n \mathrm{Hom}(\Lambda, \mathbb{Z})$. Since V is a $\mathbb{C}$-vector space, these spaces come naturally equipped with a pure Hodge structure given as follows. Recall Λ is generated by an $\mathbb{R}$-basis of V, so we have natural isomorphisms

$$H^n(X, \mathbb{C}) \cong H^n(X, \mathbb{Z} \otimes \mathbb{Z}) \cong \textstyle\bigwedge^n \mathrm{Hom}(\Lambda, \mathbb{C}) \cong \bigwedge^n \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong \displaystyle\bigoplus_{p+q=n} \textstyle\bigwedge^p V^\vee \otimes \bigwedge^q \overline{V^\vee},$$

where the final isomorphism comes from observing the decomposition $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^\vee \oplus \overline{V^\vee}$ into spaces of $\mathbb{C}$-linear and $\mathbb{C}$-anti-linear maps. Thus we declare $H^{p,q}(X) = \bigwedge^p V^\vee \otimes \bigwedge^q \overline{V^\vee}$.

Theorem 1.3 (Hodge decomposition for tori). *The spaces $H^{p,q}(T)$ form an integral Hodge structure on $H^n(T, \mathbb{Z})$.*

Hodge decomposition doesn't hold for arbitrary complex manifolds; one typically requires extra structure. When there is a Kähler metric, this structure is the presence of harmonic forms. On a torus, it is the group structure as we shall now see.

Say a smooth complex n-form ω is *invariant* if $(+y)^*\omega = \omega$ for all $y \in T$. Write $\mathrm{Inv}^n(T)$ for the space of invariant n-forms, and note that pulling back along the map $\pi : V \to T$ induces a bijection between invariant n-forms on T and linear n-forms on V (i.e. n-forms $df_1 \wedge \ldots \wedge df_n$ for $\mathbb{R}$-linear functions $f_1, \ldots, f_n : V \to \mathbb{C}$). Thus we have $\mathrm{Inv}^n(T) \cong \bigwedge^n \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$. The following proposition identifies this space with the de Rham cohomology.

Proposition 1.4. *Invariant n-forms on T are closed, and every cohomology class in $H^\bullet(T, \mathbb{C})$ can be represented by a unique invariant form, giving an isomorphism $\mathrm{Inv}^n(T) \cong H^n(X, \mathbb{C})$.*

Choosing $\mathbb{C}$-linear coordinates $z_1, \ldots, z_d \in V^\vee$, it follows that every (p, q)-class is uniquely represented by an invariant form $\sum \lambda_{IJ} dz_I \wedge d\bar{z}_J$ for scalars λ_{IJ}. In particular,

Theorem 1.5. *There is a natural isomorphism $H^{p,q}(X) \cong H^p(T, \Omega^q_T)$.*

Sketch. One can show that the holomorphic cotangent bundle of T is trivial, and hence $\Omega^p_T \cong \mathcal{O}_T \otimes H^{p,0}(T)$. This reduces to the case $p = 0$, since we have $H^q(T, \Omega^p_T) \cong H^q(T, \mathcal{O}_T) \otimes H^{p,0}(T)$. Now the inclusion $\underline{\mathbb{C}} \hookrightarrow \mathcal{O}_T$ induces a map $H^p(T, \mathbb{C}) \to H^p(T, \mathcal{O}_T)$, and one uses Fourier analysis to show this is precisely the projection $H^p(T, (C)) \twoheadrightarrow H^{p,0}(T)$. ☐

§ 2 Line bundles on tori

§ 2.1 Riemann forms. Note $H^2(T, \mathbb{Z}) = \bigwedge^2 \mathrm{Hom}(\Lambda, \mathbb{Z})$ is naturally the space of alternating integral 2-forms on Λ, or equivalently the space of alternating 2-forms $V \times V \to \mathbb{R}$ that are integer-valued on $\Lambda \times \Lambda$.

Definition 2.1. A *Riemann form* on the torus V/Λ is a Hermitian form $H : V \times V \to \mathbb{C}$ such that the alternating form $\mathrm{Im}H : V \times V \to \mathbb{R}$ is integer valued on $\Lambda \times \Lambda$.

It can be shown that Riemann forms on $T = V/\Lambda$ are uniquely determined by their imaginary part, which from the above discussion is an element of $H^2(T, \mathbb{Z})$ that satisfies the *Riemann period relation* $\mathrm{Im}(iv, iw) = \mathrm{Im}(v, w)$.

Proposition 2.2. *The space of Riemann forms is precisely the Néron–Severi group* $\mathrm{NS}(T) \subset H^2(T, \mathbb{Z})$*, i.e. the image of the first Chern map* $c_1 : \mathrm{Pic}(T) \to H^2(T, \mathbb{Z})$*. Moreover, this correspondence restricts to a bijection between ample Chern classes and positive definite Riemann forms (i.e. Riemann forms with positive definite imaginary part).*

Definition 2.3. A *polarisation* of $T = V/\Lambda$ is the choice of a positive definite Riemann form. A polarisation H is *principal* if $\det(\mathrm{Im}H|_\Lambda) = 1$, or equivalently if Λ admits a basis in which $\mathrm{Im}H$ is given by $\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)$.

Thus a complex torus is projective if and only if it admits a polarisation. Such complex tori are called *Abelian varieties.* By proposition 1.1, Abelian varieties are precisely projective group–schemes over $\mathbb{C}$. In fact, any complete group variety over $\mathbb{C}$ can be shown to be projective, and hence Abelian.

§ 2.2 The Appel–Humbert theorem. Since $\mathrm{NS}(T)$ is free for a complex torus, the exact sequence of groups $0 \to \mathrm{Pic}^0 T \to \mathrm{Pic}(T) \to \mathrm{NS}(X) \to 0$ splits (non-canonically). This can be used to give an explicit description of $\mathrm{Pic}(T)$ as follows– first note that $\mathrm{Pic}^0(T)$ is the cokernel of the map $j : H^1(T, \mathbb{Z}) \to H^1(T, \mathcal{O}_T)$ induced from the exponential exact sequence. For $T = V/\Lambda$, we have that $H^1(T, \mathcal{O}_T) = H^{0,1}(T) = \overline{V}^\vee$ and the image of j is a lattice. Thus $\hat{T} := \mathrm{Pic}^0(T)$ is also a complex torus, called the *dual* of T.

It can be shown that there is an isomorphism $\hat{T} \cong \mathrm{Hom}(\Lambda, U(1))$, i.e. the dual torus is the character-group of the period of T. One can also define the group $\mathcal{P}(T)$ of twisted semi-characters, which are given by pairs (H, α) for a Riemann form H and a function $\alpha : \Lambda \to U(1)$ satisfying $\alpha(\gamma+\delta) = \alpha(\gamma) \cdot \alpha(\delta) \cdot \exp(i\pi \cdot \mathrm{Im}H(\gamma, \delta))$.

Theorem 2.4 (Appel–Humbert). *There is an isomorphism* $\mathrm{Pic}(T) \cong \mathcal{P}(T)$ *compatible with the identifications* $\mathrm{Pic}^0(T) \cong \mathrm{Hom}(\Lambda, U(1))$ *and* $\mathrm{NS}(T) \cong \{$*Riemann forms*$\}$.

Sketch. Given a pair $(H, \alpha) \in \mathcal{P}(T)$, one uses it to describe an action $U(1) \circlearrowright \mathbb{C} \times V$. The induced map $(\mathbb{C} \times V)/U(1) \to V/\Lambda$ gives a holomorphic line bundle [see Mum85, section I.2]. □

Remark 2.5. For a point $x \in T$ in the complex torus and a line bundle $L \in \mathrm{Pic}(T)$, the bundle $(+x)^*L \otimes L^\vee$ has degree zero and hence we have a map $\phi_L : T \to \hat{T}$. This is a holomorphic homomorphism of complex tori, and the line bundle L is ample if and only if $H^0(T, L) \neq 0$ and ϕ_L is an isogeny (i.e. a surjective homomorphism).

§ 2.3 The theta divisor. The sections of the bundle associated to $(H, \alpha) \in \mathcal{P}(T)$ constructed in theorem 2.4 naturally corresond to holomorphic functions on V that are invariant under the given action of $U(1)$. This gives a functional equation, the solutions of which are called *theta-functions* of (H, α). The vanishing locus of a theta function gives a divisor corresponding to this bundle, called a *theta divisor.*

Proposition 2.6 [Mum85, p. 26]. *If* (H, α) *is a polarisation on the torus* V/Λ *, then associated space of theta functions has dimension equal to* $\sqrt{\det(\mathrm{Im}H|_\Lambda)}$.

Thus choosing a polarisation on T is equivalent to choosing an ample line bundle L, and this polarisation is principal if and only if $\dim H^0(T, L) = \chi(L) = 1$ (where we note that all higher cohomologies of L are trivial by the Kodaira vanishing theorem.)

Corollary 2.7. *Every principally polarised Abelian variety has a unique associated theta divisor up to translation.*

Proof. Suppose Θ_1, Θ_2 are two theta divisors associated to a principal polarisation (H, α), then note that the line bundle $\mathcal{O}(\Theta_1 - \Theta_2)$ has trivial Chern class. Now $L = \mathcal{O}(\Theta_2)$ is ample, hence the map $\phi_L : T \to \mathrm{Pic}^0(T)$ is an isogeny. In particular, it is surjective so there is an $x \in T$ such that $\mathcal{O}(\Theta_1 - \Theta_2) = (+x)^*\mathcal{O}(\Theta_2) \otimes \mathcal{O}(-\Theta_2)$. It follows that $\mathcal{O}(\Theta_1) = (+x)^*\mathcal{O}(\Theta_2)$, i.e. Θ_1 is linearly equivalent to $\Theta_2 + x$. But (H, α) is principal, so the associated line bundle has a unique global section (i.e. a unique corresponding divisor). Thus $\Theta_1 = \Theta_2 + x$. □

References

[Mum85] David Mumford. *Abelian Varieties.* Tata Institute of Fundamental Research, 1985 (cit. on p. 2).