Invitation to Deformation Theory

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0. Introduction

Deformation theory is a ubiquitous part of modern algebra and geometry, and is concerned with studying infinitesimal neighbourhoods of algebro-geometric objects in families that they sit in. The goal of this exposition is to provide an accessible introduction to various facets of the subject with sufficiently verbose examples, without pursuing any direction in great depth.

We begin by making the notion of a deformation problem precise in Section 1. Broadly speaking, this will be a functor that takes a base scheme and gives all deformation families of the object in concern parametrised over that base. Studying all families over a fixed base is a worthy pursuit in its own right– many theorems in algebraic geometry are proven by showing an object in concern can be deformed into something degenerate, which is hence easier to handle. One such result is Mori's theorem on existence of rational curves in Fano varieties [a treatment of this result can be found in Mat02, Section 10.1]. In Section 2 we compute such deformation families over the base $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$ as explicitly as possible. In particular, we show that smooth affine varieties are characterised by their deformation families over this base.

In Section 3, we show that the space of differential operators on a complex variety is naturally a deformation of its cotangent bundle. This is the starting point for a rich theory of D-modules, which allow us to talk about systems of differential equations on algebraic varieties.

In addition to studying individual deformation families over a specified base, we can also study properties of the deformation functor itself. For instance, the (pro) representing object for a deformation functor, if it exists, is uniquely defined and contains most of the information about the deformation problem at hand. By associating an appropriate deformation problem to a system, modern geometers are able to extract important invariants– for instance, Donovan-Wemyss's *contraction algebras* pro-represent a deformation functor associated to compoint du-Val singularities and capture most of the homological information about the singularity and its resolutions [Wem, Section 5].

In classical deformation theory (with commutative test objects), a theorem of Schlessinger characterises when the deformation functor is (pro) representable [the result is discussed in Har10, Section 16]. One such example is the deformation functor associated to a point p in a variety $X = \operatorname{Spec} A$. As one might expect it is possible to recover the local geometry of the variety near p entirely from the associated deformation theory, and we show that completed local ring $\widehat{A}_p$ pro-represents the deformation functor. In Section 4, we exhibit how extending the deformation functor to non-commutative test objects makes a similar result hold even for non-commutative algebras A, despite the loss of a geometric picture.

By introducing Deligne's philosophy of associating deformation functors to differentially graded lie algebras and A_∞-algebras, we reprove Ed Segal's result which uses deformation theory to derive a finite presentation for the non-commutative completion $\widehat{A}_p$.

Prerequisites and conventions. We assume familiarity with basic algebraic geometry and commutative algebra– at the level of a first course in scheme theory [such as Har77, chapter II] should be more than sufficient. Likewise, elementary notions in category theory and homological algebra are freely used. Parts of the exposition mention symplectic geometry and theory of differential equations, but not much will be lost if the reader has not seen these.

Unless otherwise specified, we work with an algebraically closed field k of characteristic zero (although neither of these assumptions might be strictly necessary), and nothing will be lost by assuming $k = \mathbb{C}$. A variety is an integral separated scheme of finite type over k. We do not make any distinction between a locally free sheaf and the total space of the corresponding vector bundle.

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We make no claim to originality, and while we have made an attempt to provide references throughout the text, we mention here the most helpful resources used– [Sze99; Har10; Bel18] for classical deformation theory of algebras and schemes, [Bel+16; HT07; Gin98; Oki22] for D-modules, [Aug+15] for differentially graded (lie) algebras and their relation to deformation theory, and [ELS17; Eri14; Seg08] for non-commutative deformation theory.

1. What is a deformation?

We explore how deformation problems naturally arise when considering algebro-geometric objects.

Example 1. Consider the affine scheme $X_0 = \operatorname{Spec} k[x]/(x^2)$, the double point. This can be seen as 'limiting case' of the scheme $X_t = \operatorname{Spec} k[x]/(x^2 - t)$ for $t \in k$: if $t \neq 0$ then X_t is the reduced scheme with two points. We will eventually show this is the only non-trivial way to deform X_0 as an abstract scheme, and this should agree with our intuition that the only way to deform a double point is to separate its components.

On the other hand, consider X_0 with its embedding into $\mathbb{A}^1$ as the double point at 0. Deforming X_0 in the ambient space, we have a second family of embedded subschemes X'_t given by $\mathbb{V}((x-t)^2) \subset \operatorname{Spec} k[x]$. This corresponds to deforming X_0 by translating it, without changing its intrinsic geometry. This deformation is genuinely distinct from both the trivial deformation (i.e. the constant family) and a deformation which separates the points.

Lastly we can instead consider the $k[x]$-module $M_0 = k[x]/(x^2)$, as a sheaf over $\mathbb{A}^1$. This is simply the structure sheaf of $\mathcal{O}_{X_0}$ pushed forward along the inclusion, and the family $\mathcal{O}_{X'_t}$ can be considered a deformation of M_0 as a module. Another family is given by $M_t = k^{\oplus 2}$ where x acts as $(a, b) \mapsto (bt, a)$. Observe that this cannot be realised as the structure sheaf of an embedded deformation of X_0, since M_t cannot be expressed as a quotient of $k[x]$ for $t \neq 0$.

Thus any structure we can associate multiple deformation problems– a choice which is made based on the broader context of what we are studying. To establish a common framework for such problems, we begin by making the three notions of deformations seen in Example 1 precise in a way that lends itself to functorial formulation.

1.1 Various deformation problems

We begin by outlining reasonable candidates for what one might call a 'deformation problem'– given a structure (a scheme, a sheaf, a module etc) X, we wish to find a 'nice family' containing X, and if possible classify such families up to isomorphism. The way to make this notion of a 'nice family' precise usually depends on the specific structure we wish to deform and the aspects which we wish to preserve.

Abstract deformations of schemes. Fix a k-scheme T. The category of T-schemes has objects given by diagrams $X \xrightarrow{\pi} T$ in Sch/k. We will often abuse notation and write X_T for this object. For each scheme-theoretic point $t \to T$, the fiber

$$X_t := \pi^{-1}(t) = t \times_T X_T$$

is a scheme over t, allowing us to think of X_T as a family of schemes parametrised by T. However for this interpretation to be sensible, we desire the fibers to be "similar"– this is ensured by requiring the morphism π to be flat. Accordingly, define a T*-family of schemes* is a flat morphism $X_T \xrightarrow{\pi} T$ of k-schemes.

An *augmentation* of the k-scheme T is the choice of a k-point $0 \hookrightarrow T$ (called the *special point*). Often there will be a natural choice of augmentation– for instance if T is the spectrum of a local k-algebra, there is a unique k-point corresponding to the maximal ideal.

If T is augmented and X_T is a T-scheme, the fiber X_0 over the special point is called the *special fiber*.

Definition 2. For $T, X \in \mathrm{Sch}/k$ such that T is augmented, a T*-deformation* of X is a T-family $X_T \to T$ and an isomorphism $\varphi : X \to X_0$ identifying the special fiber with X.

Often we will simply write X_T for such a deformation, leaving the morphism to T and the isomorphism φ implicit.

We say the T-deformations X_T and X'_T are isomorphic if there is an isomorphism of T-schemes $X_T \xrightarrow{\sim} X'_T$ which respects the identifications of the special fibers with X.

Embedded deformations. While we deformed schemes abstractly above, we can instead deform them inside an ambient space. For a k-scheme X, a T*-family of closed subschemes in* X is given by commutative diagram

$$\begin{array}{ccc} Z_T & \overset{\iota}{\hookrightarrow} & X \times T \\ & \searrow^{\pi} & \downarrow \mathrm{pr}_2 \\ & & T \end{array}$$

such that ι is a closed immersion and π is flat. In particular, Z_T is a T-family such that for any point $t \hookrightarrow T$ the fiber Z_t is naturally a closed subscheme of $X \times t$ via the inclusion ι.

If T is augmented, note that the special fiber is a closed subscheme of X.

Definition 3. Let T be an augmented k-scheme, and $Z \overset{i}{\hookrightarrow} X$ a closed immersion in Sch/k. A T*-deformation of* Z *in* X is a T-family $Z_T \overset{\iota}{\hookrightarrow} X \times T$ of closed subschemes in X and an isomorphism $\varphi : Z \to Z_0$ such that $\iota|_{Z_0} \circ \varphi = i$.

Deformations of sheaves. Embedded deformations are nice because we have the ambient space to work with– we are looking for quotients $\mathcal{O}_{Z_T}$ of $\mathcal{O}_{X \times T}$ which are flat over T, and give $\mathcal{O}_Z$ upon restriction to X. More generally, we can consider deformations of coherent sheaves over an augmented k-scheme T.

Definition 4. If $\mathcal{F}$ is a coherent sheaf on a k-scheme X, then a T*-deformation* of $\mathcal{F}$ is a coherent sheaf $\mathcal{F}_T$ on $X \times T$, flat over T, with an isomorphism $\varphi : \mathcal{F} \to \mathcal{F}_T|_{X \times 0}$.

Observe that embedded T-deformations of $Z \hookrightarrow X$ are precisely those deformations of the sheaf $\mathcal{O}_Z$ on X which can be expressed as quotients of the sheaf $\mathcal{O}_{X \times T}$. Example 1 exhibits that in most cases there are more deformations of the sheaf $\mathcal{O}_Z$ than there are deformations of the associated embedded subscheme.

1.2 Functoriality in the parameter space

A morphism of augmented k-schemes is a k-scheme morphism $T' \to T$ such that the natural diagram

$$\begin{array}{ccc} T' & \longrightarrow & T \\ \uparrow & & \uparrow \\ \mathrm{Spec}\, k & = & \mathrm{Spec}\, k \end{array}$$

commutes, where the vertical arrows are the augmentation morphisms. This allows us to define a category Sch^1/k (the category of k-schemes with one marked k-point), whose objects are augmented k-schemes and morphisms are of the kind described above.

If $T' \to T$ is a morphism of augmented k-schemes, and $X_T \to T$ is a T-deformation of $X \in \mathrm{Sch}/k$, then the pullback

$$\begin{array}{ccc} X_{T'} & \longrightarrow & X_T \\ \downarrow & \square & \downarrow \\ T' & \longrightarrow & T \end{array}$$

gives a T'-deformation $X_{T'}$ of X. Writing $\mathcal{D}ef_X(T)$ for the set of abstract T-deformations of X up to isomorphism, we see that we in fact have a functor $(\mathrm{Sch}^1/k)^{\mathrm{op}} \to \mathrm{Set}$.

The pullback construction works analogously for embedded deformations and deformations of sheaves.

Size of the parameter space. Global information carried by deformation families is often largely irrelevant, and we mostly care about deformations in an (analytic or infinitesimal) neighbourhood of the special point. Thus working with parameter spaces in Sch^1_k is not optimal, and we work instead with spectra of complete (Artinian in the infinitesimal case) local rings.

Write cArt^1_k for the category of commutative Artinian local k-algebras with residue field k, and $\widehat{\mathrm{cArt}}^1_k$ for the category of complete local k-algebras with residue field k. It is immediate that

$$\mathrm{cArt}^1_k \quad \subset \quad \widehat{\mathrm{cArt}}^1_k \quad \subset \quad (\mathrm{Sch}^1_k)^{\mathrm{op}}$$

as full subcategories, where the last inclusion is via Spec. Then all the deformation functors defined in Section 1.1 can be restricted to functors on cArt^1_k instead.

For $R \in \widehat{\mathrm{cArt}}^1_k$, write $0_R : R \twoheadrightarrow k$ for the augmentation morphism and $\mathfrak{m}_R$ for the maximal ideal.

1.3 Deformation theory of a point

As an extended example of the theory developed above, we study the deformation theory of a closed point in a scheme. Given a k-scheme X with a k-point x, our intuition suggests that the infinitesimal deformation theory of structures associated to x should only depend on an analytic neighbourhood of x in X. We make this notion precise by showing that the functor $\mathcal{D}ef_{x \hookrightarrow X}$ is naturally isomorphic to $\mathrm{Hom}(\widehat{\mathcal{O}}_{X,x}, -)$.

First, observe that we can focus on affine schemes in light of the following lemma.

Lemma 5. *If $U \subset X$ is an open subscheme containing x, then for any $R \in \mathrm{cArt}^1_k$ the natural map*

$$\mathcal{D}ef_{\mathcal{O}_{X,x}}(R) \to \mathcal{D}ef_{\mathcal{O}_{U,x}}(R)$$

(given by pulling back families) is a bijection. Thus the deformation theory of the skyscraper sheaf $\mathcal{O}_{X,x}$ only depends on a neighbourhood of x.

Proof. Observe that the scheme $X \times \mathrm{Spec}\, R$ has the same underlying topological space as X, but with structure sheaf $\mathcal{O}_X \otimes R$. An R-deformation of $\mathcal{O}_{X,x}$ is given by a $\mathcal{O}_X \otimes R$-module $\mathcal{F}$ such that each stalk $\mathcal{F}_p$ is a flat R-module satisfying $\mathcal{F}_p \otimes_R k \cong (\mathcal{O}_{X,x})_p$. But flat modules over Artinian local rings are free, so we must have that $\mathcal{F}$ is a skyscraper sheaf at x. Thus the restriction map

$$\begin{aligned} \mathcal{D}ef_{\mathcal{O}_{X,x}}(R) &\to \mathcal{D}ef_{\mathcal{O}_{U,x}}(R) \\ \mathcal{F} &\mapsto \mathcal{F}|_U \end{aligned}$$

admits an inverse which is extension by zero on $X \setminus U$. □

This allows us to assume X is an affine scheme $\mathrm{Spec}\, A$ for the rest of this discussion, and the point x is given by a surjection $A \twoheadrightarrow k$ which we shall also call x. Write $\mathfrak{m}_x$ for the kernel of this map.

From the above discussion, any R-deformation of $\mathcal{O}_{X,x}$ is given by an A-module structure on the free R-module of rank 1 such that the diagram below commutes.

$$(*) \qquad \begin{array}{ccc} k & \overset{0_R}{\twoheadleftarrow} & R \\ {\scriptstyle x}\uparrow & & \uparrow \\ A & \longrightarrow & A \otimes R \end{array}$$

Here $A \otimes R \to R$ is the map of A, R-bimodules given by $a \otimes r \mapsto ar$. Observe that the image of $A \otimes R \to R$ (which is an R-submodule of R, i.e. an ideal) is not contained in $\mathfrak{m}_R$. Thus the map $A \otimes R \to R$ must be surjective, and the kernel is an ideal so we have a closed immersion $\operatorname{Spec} R \hookrightarrow X \times \operatorname{Spec} R$ which sends the special point to x. This is precisely an embedded deformation of $x \hookrightarrow X$, and it is immediate that all embedded deformations must arise in this way. Thus the deformation theory of a point in X coincides with the deformation theory of its skyscraper sheaf.

We will now show that this deformation theory only depends on the completion $\widehat{A}_{\mathfrak{m}_x}$.

Proposition 6. *Let X be a k-scheme with a k-point x. Then for any $R \in cArt^1_k$, the following sets are natural bijection:*

- *(i) the set of embedded R-deformations of x in X,*
- *(ii) the set of R-deformations of the skyscraper sheaf $\mathcal{O}_{X,x}$,*
- *(iii) the set of morphisms $\operatorname{Spec} R \to X$ that send the special point $0_R \hookrightarrow \operatorname{Spec} R$ to x, and*
- *(iv) $\operatorname{Hom}(\widehat{\mathcal{O}}_{X,x}, R)$, the set of local k-algebra homomorphisms from the completion of the local ring of $\mathcal{O}_{X,x}$ to R.*

Proof. We work in the affine case. The bijection (i) $\leftrightarrow$ (ii) was shown in the discussion above.

Given a R-deformation of $\mathcal{O}_{X,x}$, we can again construct the diagram $(*)$. The composite map $A \to R$ gives a morphism $\operatorname{Spec} R \to X$ which sends the special point to x. On the other hand, given a map $A \to R$ such that the composition $A \to R \twoheadrightarrow k$ is x, we can factor the map $A \to R$ as $A \hookrightarrow A \otimes R \to R$ to get the commutative diagram of the form above, giving an R-deformation of $\mathcal{O}_{X,x}$. This is the required bijection (ii) $\leftrightarrow$ (iii).

Lastly, if $A \to R$ is a map such that the composition $A \to R \twoheadrightarrow k$ is x, then any element of $A \setminus \mathfrak{m}_A$ must land in $R \setminus \mathfrak{m}_R$ and thus has an invertible image. This gives an induced map $A_{\mathfrak{m}_x} \to R$. Since R is artinian, the map factors through $A_{\mathfrak{m}_x}/\mathfrak{m}_x^n \to R$ for some n, giving an induced map $\widehat{A}_{\mathfrak{m}_x} \to R$. Conversely, a map $\widehat{A}_{\mathfrak{m}_x} \to R$ must factor through $A_{\mathfrak{m}_x}/\mathfrak{m}_x^n \cong \widehat{A}_{\mathfrak{m}_x}/\mathfrak{m}_x^n \to R$ for some n. This gives the required map $A \to R$, and the bijection (iii) $\leftrightarrow$ (iv). □

Prorepresentability. In most cases the completed local ring $\widehat{\mathcal{O}}_{X,x}$ is not Artinian, so the functor $\mathcal{D}ef_{x \hookrightarrow X}$ is not representable. However, we do have $\widehat{\mathcal{O}}_{X,x} \in \widehat{\mathrm{cArt}}^1_k$, so that

$$\mathcal{D}ef_{x \hookrightarrow X}(R) = \varprojlim \operatorname{Hom}(\widehat{\mathcal{O}}_{X,x}/\mathfrak{m}_x^n, R),$$

where $\mathfrak{m}_x \subset \widehat{\mathcal{O}}_{X,x}$ is the maximal ideal. The inverse system $(\widehat{\mathcal{O}}_{X,x}/\mathfrak{m}_x^n)_{n \in \mathbb{N}}$ does lie in cArt^1_k, and we say the deformation functor is *prorepresented* by $\widehat{\mathcal{O}}_{X,x}$.

Given a category $\mathbf{C}$ with procategory $\widehat{\mathbf{C}}$ (i.e. $\mathbf{C}$ is a full subcategory of $\widehat{\mathbf{C}}$ and every object of $\widehat{\mathbf{C}}$ is the limit of some inverse system in $\mathbf{C}$), we can define the notion of prorepresentability of functors as above– say a functor $F : \mathbf{C} \to \mathrm{Set}$ is prorepresented by $X = \varprojlim X_n$ if there are natural bijections

$$F(Y) \cong \varprojlim \operatorname{Hom}(X_n, Y).$$

Using Yoneda's lemma, we can show that the prorepresenting object is unique if it exists.

1.4 Deformations as local moduli

Recall a *moduli functor* is a functor $(\mathrm{Sch}_k)^{\mathrm{op}} \to \mathrm{Set}$ which assigns to each k-scheme T the set of 'families of objects over T up to equivalence'. Every k-scheme X determines a moduli functor $\mathrm{Mor}(-, X)$. If a moduli functor F is naturally isomorphic to $\mathrm{Mor}(-, X)$ for some $X \in \mathrm{Sch}_k$, we say X is the *(fine) moduli space* associated to F. By Yoneda's lemma, the moduli space is unique up to unique isomorphism whenever it exists.

Note that moduli functors are completely determined by their restriction to the full subcategory of affine schemes $(\mathrm{AffSch}_k)^{\mathrm{op}} \subset (\mathrm{Sch}_k)^{\mathrm{op}}$. But this is equivalent to the category of commutative k-algebras, so we can think of moduli functors as functors $\mathrm{cAlg}_k \to \mathrm{Set}$.

Given a reasonable moduli functor $F : \mathrm{cAlg}_k \to \mathrm{Set}$ and an object $Z \in F(k)$, there is an associated deformation problem given by

$$\begin{aligned} \mathcal{D}ef_Z : \mathrm{cArt}^1_k &\to \mathrm{Set} \\ R &\mapsto \{Z_R \in F(R) \mid F(0_R)(Z_R) = Z\}. \end{aligned}$$

If the functor F is represented by the moduli space X, then by Proposition 6 we see that the deformation functor $\mathcal{D}ef_X$ gives precisely the embedded deformations of the k-point in X corresponding to Z. Then, for example, we can read off various local properties of X near this point (the dimension, reducedness, the completed local ring etc.) by simply examining the deformation functor. Indeed this is one of the motivations for studying deformation theory– even if a certain moduli space exists as scheme it is typically very hard to construct, but deformation theory allows us to study its local properties nonetheless. In cases where a moduli space doesn't exist, deformation theory provides some insight into the reasons why.

Example 7. The Hilbert functor of closed subschemes in a k-scheme X is given on objects of Sch/k by

$$\mathrm{Hilb}_{X/k}(T) = \{T\text{-families of closed subschemes in } X\},$$

and acts on morphisms by pulling back families. By a theorem of Grothendieck, this functor is representable and the moduli space $\underline{\mathrm{Hilb}}_{X/k}$ is a projective k-scheme. The k-points of $\underline{\mathrm{Hilb}}_{X/k}$ correspond to closed subschemes of X. Fix such a k-point p, corresponding to $Z \hookrightarrow X$. The associated deformation functor is given by

$$\mathcal{D}ef_{Z\hookrightarrow X}(R) = \{R\text{-deformations of } Z \text{ in } X\}.$$

2. Deformations of affine schemes

In this section we study the deformation theory of affine k-schemes (equivalently k-algebras) as explicitly as possible– first in an embedded setting, and then as abstract schemes.

First-order deformations. If $\mathcal{D}ef : \mathrm{cArt}^1_k \to \mathrm{Set}$ is any deformation functor, we say its *tangent space* is given by the set $\mathcal{D}ef(k[\epsilon]/(\epsilon^2))$. Here $k[\epsilon]/(\epsilon^2)$ is the ring of *dual numbers*. We say each element of the tangent space is a *first-order* deformation. Any $R \in \mathrm{cArt}^1_k$ admits surjections onto $k[\epsilon]/(\epsilon^2)$, so every R-deformation has to be the lift of some first-order deformation. Thus studying first-order deformations is a step towards understanding deformations over other rings, and that is the aim of this section.

All algebras considered in this section will be commutative. Abusing notation, we will write $k[\epsilon]$ for the ring of dual numbers and $R[\epsilon] = R \otimes_k k[\epsilon]$ for any k-algebra R, leaving it implicit that $\epsilon^2 = 0$. Moreover, for this section all deformations will be over $k[\epsilon]$ so we will omit the term 'first-order', referring to them simply as *deformations*.

Remark 8. The terminology 'tangent space' comes from the following observation– a k-scheme morphism $\mathrm{Spec}\, k[\epsilon] \to X$ (i.e. a $k[\epsilon]$-point in X) is given by a k-point $x \in X$ and a vector $\theta \in T_x(X)$ in the tangent space of X at x. In particular if X is the moduli space for a moduli functor F, and $\mathcal{D}ef_Z$ is the deformation functor associated to $Z \in F(k)$, then $\mathcal{D}ef_Z(k[\epsilon])$ is precisely the tangent space to X at the point corresponding to Z.

We have the following characterisation of flatness for modules over $k[\epsilon]$, which we will use repeatedly:

Lemma 9. *Let M be a $k[\epsilon]$-module. Then M is flat if and only if tensoring the exact sequence*

$$0 \to k \xrightarrow{\epsilon} k[\epsilon] \longrightarrow k \to 0$$

with M preserves exactness, where the second non-zero map is the quotient by (ϵ).

Proof. A well-known criterion for flatness [see for example Eis95, proposition 6.1] is that M is flat if and only if $\mathrm{Tor}^R_1(M, R/I) = 0$ for all finitely generated ideals $I \subset R$, for any ring R. When $R = k[\epsilon]$, the only non-trivial ideal is $I = (\epsilon)$, and $R/I \cong k$. Then since

$$0 \to M \otimes_{k[\epsilon]} k \to M \to M \otimes_{k[\epsilon]} k \to 0$$

is exact by hypothesis, we have $\mathrm{Tor}^R_1(M, k) = 0$ and M is flat. □

2.1 Embedded deformations

Fix an ambient scheme $X = \mathrm{Spec}\, S$, then a closed subscheme $Y = \mathrm{Spec}\, R$ is determined by a surjection $S \twoheadrightarrow R$ of k-algebras. In this setting, we can rephrase Definition 3 as follows.

Definition 10. Suppose we have a surjection $S \to R$ of k-algebras with kernel I. An *embedded (first-order) deformation* of R in S is an ideal $\tilde{I} \subset S[\epsilon]$ such that $\tilde{R} = S[\epsilon]/\tilde{I}$ is flat over $k[\epsilon]$, and such that the map $S[\epsilon] \to S$ carries $\tilde{I}$ onto I.

In fact over $k[\epsilon]$, embedded deformations of $S \twoheadrightarrow R$ are equivalent to deformations of the ideal I in a sense that we make precise below, and we will use the two notions interchangeably.

Lemma 11. *The following are equivalent.*

1. *$\tilde{I} \subset S[\epsilon]$ is an ideal that is flat over $k[\epsilon]$, and the isomorphism $S[\epsilon] \otimes_{k[\epsilon]} k \cong S$ restricts to an isomorphism $\tilde{I} \otimes_{k[\epsilon]} k \cong I$.*
2. *$S[\epsilon]/\tilde{I}$ is an embedded deformation of $S \twoheadrightarrow R = S/I$.*

Proof. (1. $\Rightarrow$ 2.) Tensoring with $k[\epsilon]/(\epsilon)$ takes $S[\epsilon] \to S$ and $\tilde{I} \to I$, so we get a map $S[\epsilon]/\tilde{I} \to S/I \cong R$ such that $(S[\epsilon]/\tilde{I}) \otimes_{k[\epsilon]} k \cong R$. To prove flatness of $S[\epsilon]/\tilde{I}$ over $k[\epsilon]$, we consider the following nine-term commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \xrightarrow{\epsilon} & \tilde{I} & \xrightarrow{\epsilon\mapsto 0} & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S & \xrightarrow{\epsilon} & S[\epsilon] & \xrightarrow{\epsilon\mapsto 0} & S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S/I & \xrightarrow{\epsilon} & S[\epsilon]/\tilde{I} & \xrightarrow{\epsilon\mapsto 0} & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Clearly the columns are exact, and the middle row is exact by flatness of S over k. The top row is right-exact a priori, as we get it by tensoring the exact sequence

$$0 \to k \xrightarrow{\epsilon} k[\epsilon] \longrightarrow k \to 0$$

with $\tilde{I}$. But it is also left exact, because if $\epsilon x = 0$ for some $x \in I$, then since $\epsilon x \in S[\epsilon]$, we must have $x = 0$. Thus by the nine-lemma, the bottom row is exact, so that $S[\epsilon]/\tilde{I}$ is flat over $k[\epsilon]$ by Lemma 9.

The proof of (2. $\Rightarrow$ 1.) is analogous, using the exactness of the bottom row to deduce the exactness of the top row. □

The following result gives a useful classification of embedded deformations.

Theorem 12. *The embedded deformations of R in S are in natural one-to-one correspondence with the set $\mathrm{Hom}_S(I, R)$ of S-module homomorphisms, with the trivial embedded deformation corresponding to the zero map.*

Proof. Suppose we have an S-module homomorphism $\phi : I \to R$. Then define an ideal $I_\phi \subset S[\epsilon]$ by

$$I_\phi = \left\{ x + \epsilon y \,\middle|\, x \in I, y \in S, \phi(x) = y \bmod I \right\}$$

I_ϕ is clearly an ideal, as for all $x + \epsilon y \in I_\phi$ and $a + \epsilon b \in S$, we have that $(x + \epsilon y)(a + \epsilon b) = xa + \epsilon(xb + ay)$. Then since $\phi(x) = y \in R$, we have $\phi(xa) = a\phi(x) = ay = ay + xb$ in R, so $(x + \epsilon y)(a + \epsilon b) \in I_\phi$.

Clearly the map $S[\epsilon] \to S$ given by dividing out by ϵS sends I_ϕ to I. It remains to show that $S[\epsilon]/I_\phi$ is flat over $k[\epsilon]$. For this, just replace $\tilde{I}$ with I_ϕ in the nine-term commutative diagram on the previous page, and then flatness follows by identical reasoning. Note that the zero map $\phi = 0 \in \hom SIR$ gives the ideal $I \oplus \epsilon I$, which is the trivial embedded deformation.

Conversely, suppose we have an embedded deformation $\tilde{I} \subset S[\epsilon]$. To define our $I \to R$, take any $x \in I$ and lift to an element of $\tilde{I}$. Since $S[\epsilon] \cong S \oplus S\epsilon$, and we can write this lift as $x + \epsilon y$ for some $y \in S$. This lift is not unique, but differs by an element of I: if $x + \epsilon y, x + \epsilon y'$ are two lifts, then $\epsilon(y - y') \in \epsilon I$. Replacing I_ϕ with $\tilde{I}$ in the previous nine-term commutative diagram, we see that since $S[\epsilon]/\tilde{I}$ is flat over $k[\epsilon]$,

$$0 \longrightarrow I \longrightarrow \tilde{I} \longrightarrow I \longrightarrow 0$$

is exact, so that $y - y' \in I$. Thus, the lift y is unique upto an element of I, giving a well-defined S-homomorphism $I \to R$. Note that the trivial deformation $I[\epsilon] = I \oplus I\epsilon$ gives the 0 element in $\hom SIR$, as for any $x \in I$ and lift $x + \epsilon y$, we have that $y \in I$, so is zero in R.

It remains to show that these two correspondences are inverse to one another. So given an S-homomorphism $\phi : I \to R$, we get an embedded deformation I_ϕ. Then the element of $\mathrm{Hom}_S(I, R)$ induced by I_ϕ is given as follows: fix $x \in I$, then choose any lift $x + \epsilon y \in I_\phi$ for some $y \in S$. The induced map is then $x \mapsto y \in R$, which is equal to $\phi(x)$. Thus, the homomorphism induced by I_ϕ is the same as ϕ.

Conversely if we have an embedded deformation $S[\epsilon]/\tilde{I}$, then let $\phi : I \to R$ be the corresponding homomorphism. We claim that $\tilde{I} = I_\phi$. Indeed if $x + \epsilon y \in \tilde{I}$, then $x \in I$ and by the definition of ϕ we have that $\phi(x) = y + I$, so that $x + \epsilon y \in I_\phi$. On the other hand if $x + \epsilon y \in I_\phi$, then $\phi(x) = y + I$, so that $x + \epsilon y \in \tilde{I}$. □

Remark 13. In fact, any map $I \to R$ must factor through the quotient $I \twoheadrightarrow I/I^2$, giving us natural isomorphisms

$$(\dagger) \qquad \mathrm{Hom}_S(I, R) \cong \mathrm{Hom}_S(I/I^2, R) \cong \mathrm{Hom}_R(I/I^2, R).$$

This admits the following geometric interpretation– whenever $Y \hookrightarrow X$ is a closed subscheme defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, we can define the *normal sheaf* of Y in X as

$$\mathcal{N}_{Y/X} := \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y).$$

In reasonable cases (eg. when Y is a local complete intersection in X), this is a vector bundle (the normal bundle). Then Theorem 12 along with (†) says that (in the affine case) embedded deformations of $Y \hookrightarrow X$ are given by global sections of $\mathcal{N}_{Y/X}$. Thus to deform Y in X, we must 'perturb it in normal directions'.

Deformations in codimension 1 have a particularly nice characterisation– observe that if $f \in S$ is not a zero-divisor then the ideal (f) is isomorphic to S as an S-module, and hence deformations of $S \twoheadrightarrow S/(f) = R$ are in bijection with elements of R. Moreover, the deformations of principal ideals can be explicitly given by choosing suitable lifts of elements in R as follows.

Corollary 14. *Suppose* $I \subset S$ *is a principal ideal with generator* f. *Then the embedded deformation of* $S \twoheadrightarrow S/I = R$ *corresponding to* $\phi \in \mathrm{Hom}_S(I, R)$ *is given by principal ideal* $(f + \epsilon f') \subset S[\epsilon]$, *where* $f' \in S$ *is an element such that* $\phi(f) = f' \mod I$.

Proof. In the notation of Theorem 12, if $\phi(f) = f'$ in R then $(f+\epsilon f') \subseteq I_\phi$. To show equality holds, suppose $x+\epsilon y \in I_\phi$. Then $x = \alpha f$ for some $\alpha \in S$, and $\alpha\phi(f) = y \in I$. Thus, $y = \alpha\phi(f) + \beta f$ for some $f \in S$, so that

$$\begin{aligned} x + \epsilon y &= \alpha f + \epsilon(\alpha\phi(f) + \beta f) \\ &= (\alpha + \epsilon\beta)f + \epsilon\alpha\phi(f) \\ &= (\alpha + \epsilon\beta)(f + \epsilon\phi(f)). \end{aligned}$$

Then since $f + \epsilon\phi(f) \in (f + \epsilon f')$, we have that $x + \epsilon y \in (f + \epsilon f')$. □

Using Theorem 12 and Corollary 14, we can now compute many examples of embedded deformations.

Example 15 (Closed subschemes of $\mathbb{A}^1_k$). The ring $S = k[x]$ is a principal ideal domain, so any proper ideal I is generated by some polynomial f of degree $n \geqslant 0$. Writing $R = S/I$, we see that the closed subscheme $\mathrm{Spec}\, R \hookrightarrow \mathrm{Spec}\, S$ consists of n points on $\mathbb{A}^1_k$, counted with multiplicity. Then by Theorem 12, the space of embedded deformations is in bijection with $k[x]/(f)$. But this is just the set of all polynomials in $k[x]$ of degree less than n, so by Corollary 14 all the embedded deformations of R are of the form

$$k[\epsilon, x]/(f + \epsilon(a_{n-1}x^{n-1} + \ldots + a_1 x + a_0))$$

for $a_0, \ldots, a_{n-1} \in k$. This naturally has the structure of an n-dimensional vector space, thus showing that the Hilbert scheme of n points in $\mathbb{A}^1_k$ is n-dimensional.

2.2 Abstract deformations

Again, Definition 2 can be rephrased for first-order deformations of algebras as follows.

Definition 16. An *abstract deformation* of a k-algebra R is a flat $k[\epsilon]$-algebra $\widetilde{R}$ together with a surjective k-algebra homomorphism $\beta : \widetilde{R} \to R$ such that the induced map

$$\widetilde{R} \otimes_{k[\epsilon]} k \longrightarrow R$$

is an isomorphism.

Two such deformations $\beta : \widetilde{R} \to R$ and $\beta' : \widetilde{R}' \to R$ are equivalent if there is $k[\epsilon]$-algebra homomorphism $\alpha : \widetilde{R} \to \widetilde{R}'$ such that the functor $- \otimes_{k[\epsilon]} k$ takes α to id_R. More precisely, there is a commutative diagram of k-algebras:

$$\begin{array}{ccc} \widetilde{R} \otimes_{k[\epsilon]} k & \xrightarrow{\alpha \otimes \mathrm{id}} & \widetilde{R}' \otimes_{k[\epsilon]} k \\ \downarrow \cong & & \downarrow \cong \\ R & =\!= & R \end{array}$$

It is immediate that $\alpha : \widetilde{R} \to \widetilde{R}'$ is an isomorphism by applying the five lemma to the following diagram with exact rows.

$$\begin{CD}
0 @>>> R @>{\epsilon}>> \widetilde{R} @>{\beta}>> R @>>> 0 \\
@. @| @VV{\alpha}V @| @. \\
0 @>>> R @>{\epsilon}>> \widetilde{R}' @>{\beta'}>> R @>>> 0
\end{CD}$$

The *trivial deformation* of a k-algebra R is $R \otimes_k k[\epsilon] \cong R[\epsilon]$. (To see this is indeed a deformation, observe that flatness is preserved by base change and R is automatically flat over k, so $R[\epsilon]$ is flat over $k[\epsilon]$. The equality $(R \otimes_k k[\epsilon]) \otimes_{k[\epsilon]} k \cong R$ follows from standard commutative algebra.) We will say that a deformation is *trivial* if it is equivalent to the trivial deformation, and *non-trivial* otherwise.

Abstract deformations can be realised in an ambient scheme. If we have a surjection $S \twoheadrightarrow R$, each associated embedded deformation $S[\epsilon] \twoheadrightarrow \widetilde{R}$ gives rise to an abstract deformation $\widetilde{R}$ of R. In fact for a sufficiently large ambient scheme, all abstract deformations can be realised as embedded deformations in this way.

Proposition 17. *There is a* k*-algebra surjection* $S \twoheadrightarrow R$ *such that whenever* $\beta : \widetilde{R} \to R$ *is an abstract deformation of* R*, there is a commutative diagram*

$$\begin{CD}
S[\epsilon] @>{\epsilon \mapsto 0}>> S \\
@VVV @VVV \\
\widetilde{R} @>{\beta}>> R
\end{CD}$$

where all the maps are surjections. In particular, $S[\epsilon] \twoheadrightarrow \widetilde{R}$ *is an embedded deformation of* $S \twoheadrightarrow R$.

Proof. Let S be any polynomial algebra $k[x_i]$ (in possibly infinitely many variables) that surjects onto R. Abusing notation, we will write x_i for the image of x_i in R.

For each i, we can choose a lift $y_i \in \widetilde{R}$ of $x_i \in R$. Since $S[\epsilon]$ is a free $k[\epsilon]$-algebra, we get a map $S[\epsilon] \to \widetilde{R}$ sending $x_i \mapsto y_i$. It remains to show this is a surjection– this follows from the weak five lemma applied to the commutative diagram below with exact rows.

$$\begin{CD}
0 @>>> S @>{\epsilon}>> S[\epsilon] @>>> S @>>> 0 \\
@. @VVV @VVV @VVV @. \\
0 @>>> R @>{\epsilon}>> \widetilde{R} @>>> R @>>> 0.
\end{CD}$$

(Surjectivity of the peripheral arrows implies surjectivity of the middle arrow). ☐

Remark 18. There is no canonical embedded deformation inducing a given abstract deformation, as we had to make a choice of lift of $x_i \in R$ to $y_i \in \widetilde{R}$. A different choice of lift, say y_i', would also give an embedded deformation $S[\epsilon] \to \widetilde{R}$ sending $x_i \mapsto y_i'$. But using the exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot\epsilon} \widetilde{R} \longrightarrow R \longrightarrow 0$$

we deduce that for every i there exists a unique $z_i \in R$ such that $y_i' = y_i + \epsilon z_i$. Thus, for any two surjections $\phi, \phi' : S[\epsilon] \to \widetilde{R}$, there is a unique $k[\epsilon]$-algebra automorphism $\alpha : \widetilde{R} \to \widetilde{R}$ sending $y_i \mapsto y_i + \epsilon z_i$ such that such that $\phi' = \alpha \circ \phi$.

Classification of abstract deformations. In light of Proposition 17, to classify abstract deformations of R it suffices to find a polynomial algebra S that surjects onto R and then determine which embedded deformations of $S \twoheadrightarrow R$ are equivalent as abstract deformations. We shall prove that two embedded deformations given by $\phi, \phi' \in \mathrm{Hom}_S(I, R)$ are equivalent if and only if the difference $\phi - \phi'$ is induced by a k-derivation $S \to R$ in the sense that we define below.

Definition 19. For a k-algebra S and an S-module M, a k-*derivation* from S into M is a k-linear map $\delta : S \to M$ which satifies the Leibniz rule– for all $s, s' \in S$,

$$\delta(ss') = s\delta(s') + s'\delta(s).$$

Write $\mathrm{Der}_k(S, M)$ for the space of k-derivations from S into M, this is naturally an S-module.

In our case, the surjection $S \to R = S/I$ induces a natural S-module structure on R. Then we can check that the restriction of a k-derivation $\delta : S \to R$ to $I \subset S$ is an element of $\mathrm{Hom}_S(I, R)$, thus inducing an embedded deformation by Theorem 12. In fact, we show that these are precisely those embedded deformations that are trivial as abstract deformations.

Theorem 20. *If $R \cong S/I$ as k-algebras, then for any $\phi, \phi' \in \mathrm{Hom}_S(I, R)$ the corresponding embedded deformations are equivalent if and only if $\phi' - \phi$ is the restriction of a k-derivation $S \to R$.*

Proof. Write $I_\phi, I_{\phi'} \subset S[\epsilon]$ for the ideals of embedded deformations corresponding to ϕ, ϕ' respectively.

Suppose $\delta : S \to R$ is a k-derivation such that $\phi'(x) - \phi(x) = \delta(x)$ for all $x \in I$. Then define a map

$$\begin{aligned} S[\epsilon] &\longrightarrow \widetilde{R}' \\ x + \epsilon y &\longmapsto x + \epsilon y + \epsilon \cdot \delta(x) \mod I_{\phi'} \end{aligned}$$

where we note that $\epsilon \cdot \delta(x)$ is the image of $\delta(x) \in R$ in the injection $R \xrightarrow{\epsilon} \widetilde{R}'$. It is clear that this is a homomorphism of k-algebras, and I_ϕ lies in the kernel since the elements of I_ϕ are of the form $x + y\epsilon \in I_\phi$ where $x \in I$ and $\phi(x) = y \mod I$. Thus we have an induced homomorphism $\alpha : \widetilde{R} \to \widetilde{R}'$. Moreover, it is clear from the construction that the functor $- \otimes_{k[\epsilon]} k$ (which essentially sets $\epsilon = 0$) sends α to Id_k, thus $\widetilde{R}$ and $\widetilde{R}'$ are equivalent as abstract deformations.

Conversely, suppose we have an equivalence of deformations $\alpha : \widetilde{R} \to \widetilde{R}'$, then consider the maps

$$S[\epsilon] \xrightarrow{\pi} \widetilde{R} \xrightarrow{\alpha} \widetilde{R}', S[\epsilon] \xrightarrow{\pi'} \widetilde{R}'.$$

These agree upon applying the functor $- \otimes_{k[\epsilon]} k$, so the image of the difference $(\pi \circ \alpha - \pi')$ must lie in $\epsilon R \subset \widetilde{R}'$. Thus we can define the composite map

$$\delta : \quad S \xrightarrow{1 \mapsto 1} S[\epsilon] \xrightarrow{\alpha \circ \pi - \pi'} \epsilon R \xrightarrow{\epsilon \mapsto 1} R.$$

It can be checked that this is a k-derivation. For $x \in I$ we have $\pi(x) + \epsilon\phi(x) = 0$ by construction of the ideal I_ϕ, and likewise $\pi'(x) + \epsilon\phi'(x) = 0$. Moreover, $\alpha(\epsilon\phi(x)) = \epsilon\phi(x)$, so $\delta = \phi' - \phi$ as required. ☐

Given a k-algebra R, choose a surjection $S \twoheadrightarrow R$ with kernel I such that S is a polynomial algebra. Note any R-module M has a naturally induced S-module structure, and the ideal I lies in the annihilator of M. Then a k-derivation $\delta : S \to M$ restricts to a homomorphism $I \to M$ since

$$\delta(s \cdot i) = s \cdot \delta(i) + i \cdot \delta(s) = s \cdot \delta(i)$$

whenever $s \in S$, $i \in I$. We define the R-module $T^1(R/k, M)$ as the cokernel of the map $\mathrm{Der}_k(S, M) \to \mathrm{Hom}_S(I, M)$ given by restriction, and write $T^1_{R/k}$ for $T^1(R/k, R)$.

Then the result below follows from Proposition 17 and Theorem 20.

Corollary 21. *There is a natural one-to-one correspondence between equivalence classes of abstract deformations of R and $T^1_{R/k}$.*

Remark 22. A priori there is no reason to believe that $T^1(R/k, M)$ is independent of the chosen embedding $S \twoheadrightarrow R$. But since it turns out to be naturally in bijection with the set of abstract deformations of R, it is indeed independent of S. In fact, T^1 is a functor which can be defined in a more general setting without making choices [see for instance Har10].

2.3 Geometric examples

If S is a polynomial algebra surjecting onto R, then we showed how the elements of $\mathrm{Hom}_S(I, R)$ naturally correspond to global sections of the normal sheaf of the inclusion Spec R ↪ Spec S, and thus deformations of Spec R come from perturbing it in normal directions in a sufficiently large ambient scheme. Which of these are trivial?

Recall that the module of Kähler differentials $\Omega_{S/k}$ gives the global sections of the cotangent sheaf on Spec(S), and is generated over S by all formal symbols $\{ds \mid s \in S\}$. The map $d : S \to \Omega_{S/k}$ given by $s \mapsto ds$ is a derivation, and is universal in the sense that for any S-module M, all k-derivations $S \to M$ must factor through d giving a natural isomorphism $\mathrm{Hom}_S(\Omega_{S/k}, M) \cong \mathrm{Der}_k(S, M)$. In particular, elements of $\mathrm{Der}_k(S, S)$ are precisely the global sections of the tangent sheaf (i.e. vector fields) on Spec(S).

When S is a polynomial algebra and $R \cong S/I$, the module $\Omega_{S/k}$ is free over S [Eis95, Corollary 16.1] and hence we have the exact sequence

$$0 \to \mathrm{Der}_k(S, I) \to \mathrm{Der}_k(S, S) \to \mathrm{Der}(S, R) \to 0.$$

Geometrically, this says that k-derivations $\delta : S \to R$ are given by vector fields on Spec S up to fields which vanish on Spec R (which naturally give sections of the normal sheaf). The induced embedded deformation perturbs Spec R along this vector field while preserving the inherent geometry, thus giving a trivial abstract deformation.

Example 23 (Zero dimensional schemes)**.** Consider $S = k[x]$ and $R = k[x]/(x)$, i.e. the inclusion of the origin in $\mathbb{A}^1_k$. We have that $\mathrm{Der}_k(S, R)$ is a one dimensional vector space with basis $\partial\partial x$, and the embedded deformation corresponding to the vector field $a\frac{\partial}{\partial x}$ is $k[\epsilon, x]/(x + a\epsilon)$ which 'translates the point with speed a'. In this case the normal bundle (i.e. the tangent space to $\mathbb{A}^1_k$ at the origin) has rank 1 and thus every embedded deformation of $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} S$ is of the form above, showing that Spec R has no non-trivial abstract deformations. We say that Spec R is *infinitesimally rigid.*

Consider instead the ordinary double point coming from $R = k[x]/(x^2)$, with the natural surjection from $S = k[x]$. A k-derivation $\delta : S \to R$ given by $\delta x = ax + b \in R$ must have $\delta(x^2) = 2bx$ and hence the induced deformation is given by the ideal $(x^2 - 2bx\epsilon) = (x - b\epsilon)^2$. This is the trivial deformation corresponding to translation along the vector field $b\frac{\partial}{\partial x}$. Note however that $\mathrm{Hom}_S(I, R)$ is a two dimensional vector space, hence an embedded deformation is given by an ideal $(x^2 - 2bx\epsilon + c\epsilon) \subset k[\epsilon, x]$ for $b, c \in k$. Thus we have a one-dimensional family of nontrivial abstract deformations

$$T^1_{R/k} = \left\{ \frac{k[\epsilon, x]}{(x^2 - c\epsilon)} \ \mid \ c \in k \right\}$$

which can be seen as separating the double point into two points moving away.

The discussion above readily applies to the setting of Example 15– suppose we have a set of p zero-dimensional subschemes, the ith one having multiplicity m_i. This can be realised as the vanishing locus of a degree $n = \sum_i m_i$ polynomial $f = \prod_i (x - \lambda_i)^{m_i} \in k[x]$ for $\lambda_i \in k$ distinct. We saw that this embedding admits an n-dimensional deformation space parametrised by polynomials of degree at most $n - 1$. We should expect the trivial deformations to form a p dimensional subspace coming from translating the 'clusters' without changing multiplicity. This is indeed the case– note that

$$h := \gcd\left(f, \frac{\partial f}{\partial x}\right) = \prod_{i=1}^{p} (x - \lambda_i)^{m_i - 1}$$

is a degree $n - p$ polynomial, and by Bézout's lemma there exist $\alpha, \beta \in k[x]$ with $h = \alpha f + \beta \frac{\partial f}{\partial x}$.

Given an embedded deformation $(f + \epsilon g) \subset k[x, \epsilon]$, we can write $g = q \cdot h + r$ for a polynomial $r \in k[x]$ with $\deg(r) < \deg(h)$. Thus we have

$$g - r = qh \equiv q\beta \frac{\partial f}{\partial x} \pmod{f},$$

i.e. the deformations $(f + \epsilon g)$ and $(f + \epsilon r)$ differ by a trivial deformation induced by the derivation $q\beta\frac{\partial}{\partial x}$. Moreover, in this case the map $(f) \to R$ given by $f \mapsto r \pmod{f}$ is a derivation if and only if $r = 0$, by degree considerations. Thus we have a complete description of the equivalence classes of deformations of $\operatorname{Spec} k[x]/(f)$ as

$$T^1_{R/k} = \left\{ \frac{k[\epsilon, x]}{f + \epsilon r} \mid \deg(r) < n - p \right\} \cong k^{n-p}.$$

Example 24 (Coordinate axes)**.** Consider the ring $R = k[x, y]/(xy)$. The natural surjection from $S = k[x, y]$ gives an embedding of Spec R as the coordinate axes in $\mathbb{A}^2_k$. Then it is clear that embedded deformations of Spec R are given by ideals $(xy + \epsilon(a + xp(x) + yq(y))) \subset k[x, y, \epsilon]$ for polynomials p, q in x, y respectively. The map

$$\begin{aligned} \frac{k[x, y, \epsilon]}{(xy + a\epsilon)} &\to \frac{k[x, y, \epsilon]}{(xy + (a + xp(x) + yq(y))\epsilon)} \\ x &\mapsto x + \epsilon q(y) \\ y &\mapsto y + \epsilon p(x) \end{aligned}$$

is easily seen to be an isomorphism of abstract deformations, induced by the derivation $\delta := q(y)\frac{\partial}{\partial x} + p(x)\frac{\partial}{\partial y}$. Moreover for distinct $a, a' \in k$, the corresponding deformations can't be equivalent because there is no way to differentiate xy and get $a - a'$. Hence the space of abstract deformations is one dimensional, given by the family of conics

$$T^1_{R/k} = \left\{ \frac{k[\epsilon, x, y]}{(xy + \epsilon a)} \mid a \in k \right\}.$$

Example 25 (Smoothening of hypersurfaces). Generalising the above example, consider $S = k[x_1, ..., x_n]$ with a principal ideal $I = (f)$. Then $R = S/I$ is the coordinate ring of a hypersurface, and the embedded deformations of $\operatorname{Spec} R$ are given by $\operatorname{Hom}_S(I, R) \cong R$. To characterise trivial deformations, note $\Omega_{S/k} = \bigoplus_{i=1}^n S \cdot dx_i$ is free and hence k-derivations $S \to R$ are of the form $\sum_{i=1}^n g_i \frac{\partial}{\partial x_i}$ for $g_i \in R$. Thus the set of trivial deformations is precisely the *Jacobian ideal*

$$\left\{ \sum_{i=1}^n \frac{\partial f}{\partial x_i} g_i \mid g_i \in R \right\} = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right) =: (\nabla f) \subset R.$$

It follows that the abstract deformations of $\operatorname{Spec} R$ are given by

$$T^1_{R/k} = \frac{R}{(\nabla f)} \cong \frac{S}{(f, \nabla f)}$$

which can be recognised as the coordinate ring of the singular locus of $\operatorname{Spec} R$, often called the *Tjurina algebra* of the hypersurface.

Rigidity of smooth affine varieties. The example above shows smooth hypersurfaces are infinitesimally rigid. We shall see in fact that this characterises smooth affine varieties.

Suppose $\operatorname{Spec} R$ is an irreducible affine scheme of finite type embedded in $\mathbb{A}^n_k$ via a surjection from $S = k[x_1, ..., x_n]$ with kernel I. Since $\mathbb{A}^n_k$ is non-singular, we see from [Har77, Theorem 8.17] that $\operatorname{Spec} R$ is non-singular if and only if $\Omega_{R/k}$ is a locally free R-module, and the map α in the conormal sequence

$$I/I^2 \xrightarrow{\alpha} \Omega_{S/k} \otimes_S R \to \Omega_{R/k} \longrightarrow 0$$

is injective. We will see that the functor T^1 provides a cohomology which measures the failure of the conormal sequence to be left exact– in particular detecting if $\operatorname{Spec} R$ is non-singular.

Proposition 26. *For any R-module M there is an exact sequence given by*

$$\operatorname{Hom}_R(\Omega_{S/k} \otimes_S R, M) \to \operatorname{Hom}_R(I/I^2, M) \to T^1(R/k, M) \to 0.$$

Proof. By the $\otimes - \operatorname{Hom}$ adjunction, we have natural isomorphisms

$$\operatorname{Hom}_R(\Omega_{S/k} \otimes_S R, M) \cong \operatorname{Hom}_S(\Omega_{S/k}, \operatorname{Hom}_R(R, M)) \cong \operatorname{Der}_k(S, M)$$

where the second isomorphism follows from the definition of $\Omega_{S/k}$ and the fact that $\operatorname{Hom}_R(R, -)$ is the identity functor. Likewise, we have

$$\operatorname{Hom}_R(I/I^2, M) \cong \operatorname{Hom}_S(I/I^2, M) \cong \operatorname{Hom}_S(I, M)$$

since I lies in the annihilator of both M and I/I^2. It can be checked then that the map induced by α is then simply given by restricting the derivation, so that the result follows. □

Theorem 27. *Let $X = \operatorname{Spec} R$ be an irreducible affine scheme of finite type over k. Then X is non-singular if and only if $T^1(R/k, M) = 0$ for all R-modules M, in which case it has no non-trivial deformations.*

Proof. Since $\Omega_{R/k}$ is finitely presented, it is locally free if and only if it is projective. From the discussion above, it follows that X is non-singular if and only if the sequence below splits:

$$0 \to I/I^2 \xrightarrow{\alpha} \Omega_{S/k} \otimes_S R \to \Omega_{R/k} \to 0.$$

Now if the sequence splits and M is an R-module, then the sequence remains split-exact after applying the functor $\operatorname{Hom}_S(-, M)$ and hence $T^1(R/k, M) = 0$.

Conversely if $T^1(R/k, M) = 0$ for all R-modules M, then choosing $M = I/I^2$ we see that

$$\operatorname{Hom}_R(\Omega_{S/k} \otimes_S R, I/I^2) \to \operatorname{Hom}_R(I/I^2, I/I^2)$$

is surjective, so there exists a $\beta : \Omega_{S/k} \otimes_S R \to I/I^2$ such that $\beta \circ \alpha = \operatorname{Id}_{I/I^2}$. This gives a splitting of the sequence. □

3. Deformation quantisation and D-modules

The deformations examined in Section 2 preserve commutativity in order to provide a geometric picture. Instead we could also deform a commutative k-algebra A to obtain something possibly non-commutative, the so-called process of *quantisation.* Classifying flat families of associative k-algebras over augmented Artinian rings naturally leads to the theory of Hochschild cohomology of the special fiber, as is discussed in [Sze99] and [Bel18].

In this section we focus on the special case of *filtered deformations* of graded rings where the deformation is naturally endowed with the structure of a Poisson algebra. As a central example, we introduce the theory of D-modules, following the exposition in [HT07] with occasional references to [Gin98] and [Bel+16].

This theory has its origins in physics, and we retain some of the conventions by working over the complex numbers and writing $\hbar$ for the indeterminate. Thus our deformations are over the powerseries ring $\mathbb{C}[\![\hbar]\!]$ and its Artinian quotients $\mathbb{C}[\hbar]/(\hbar^n)$.

3.1 The algebra of differential operators

Let X be a smooth algebraic variety over $\mathbb{C}$ of dimension n. Write $\mathcal{O}_X$ for the structure sheaf on X, and TX for the tangent sheaf of X, i.e. the sheaf of $\mathbb{C}$-linear derivations on $\mathcal{O}_X$. Then we construct the sheaf of *differential operators* on X as the subalgebra of the endomorphisms $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and TX.

More concretely on any open affine neighbourhood $U \in X$ we can choose regular functions $x_1, ..., x_n \in \mathcal{O}_X(U)$ which satisfy

$$\mathcal{O}_U = \mathbb{C}[x_i, \dots, x_n], \quad TU = \bigoplus_{i=1}^{n} \mathcal{O}_U \cdot \partial_i,$$

where the variable ∂_i corresponds to the derivation satisfying $\partial_i(x_j) = \delta_{ij}$. We call $\{x_i, \partial_i\}$ a *local coordinate system* on U. Locally the algebra of differential operators is then the algebra

$$D_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha,$$

where we write $\partial^\alpha := \prod_{i=1}^n \partial_i^{\alpha_i}$ for tuples $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$.

Note that for $f \in \mathcal{O}_U$, a straightforward application of the Leibniz rule gives us

$$\partial_i(x_j \cdot f) - x_j \cdot \partial_i(f) = \delta_{ij} \cdot f.$$

It can be checked that these are the only independent relations in D_U, giving us a presentation

$$D_U = \frac{\mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n]}{([\partial_i, x_j] - \delta_{ij})}$$

where $[\partial_i, x_j] = \partial_i x_j - x_j \partial_i$ is the commutator. In particular this algebra is noncommutative.

It is straightforward to check that these local constructions are compatible across affine patches, together giving a sheaf D_X of $\mathcal{O}_X$-algebras.

Order filtration. For a differential operator (i.e. a local section of D_X) ∂^α given by the n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$, we define its order to be $|\alpha| = \sum_{i=1}^n \alpha_i$. On any affine patch U, this gives a natural *order filtration* F of the algebra D_U via

$$F: \quad 0 = F_{-1}D_U \subset F_0 D_U \subset F_1 D_U \subset F_2 D_U \subset \dots,$$

where for $\ell \in \mathbb{N}$ we define the subspaces $F_\ell D_U \subset D_U$ as

$$F_\ell D_U = \bigoplus_{|\alpha| \leqslant \ell} \mathcal{O}_U \partial^\alpha.$$

It is clear from the definition that $F_0 = \mathcal{O}_U$ is the ring of regular functions on U. Moreover the filtration is ascending, i.e. the containment $F_\ell D_U \subsetneq F_{\ell+1} D_U$ is strict. Note also that it is compatible with multiplication in the sense that

$$\forall m, n \geqslant -1, \quad (F_m D_U) \cdot (F_n D_U) = F_{m+n} D_U.$$

Less immediate, but very useful fact about this filtration is given by the following proposition.

Proposition 28. *Let* F *be the order filtration of* D_U *and consider differential operators* $P \in F_m D_U$, $Q \in F_n D_U$. *The commutator* $[P, Q]$ *satisfies* $[P, Q] \in F_{m+n-1} D_U$.

Proof. Follows by induction with base case given by taking $\partial_i \in F_1 D_U$ and $x_j \in F_0 D_U$. As we have seen before $[\partial_i, x_j] = \delta_{ij} \in F_0 D_0 = F_{1+0-1} D_U$ as required. □

Associated graded ring and principal symbols. We can use the order filtration F of the algebra D_U to define its *associated graded ring* as

$$\mathrm{gr}^F D_U = \bigoplus_{\ell=0}^{\infty} F_\ell D_U / F_{\ell-1} D_U.$$

Multiplication in $\mathrm{gr}^F D_U$ is defined as follows: given elements $a' \in F_m D_U / F_{m-1} D_U$ and $b' \in F_n D_U / F_{n-1} D_U$ we choose representatives $a \in F_m D_U$ and $b \in F_n D_U$ and set $a'b' \in F_{m+n} D_U / F_{m+n-1} D_U$ to be the equivalence class of $ab \in F_{m+n} D_U$. One can check that this is well defined modulo $F_{m+n-1} D_U$. For a general filtered algebra, the associated graded algebra is defined analogously.

A nice consequence of Proposition 28 is that the ring $\mathrm{gr}^F D_U$ is commutative. This allows for techniques from commutative algebra to be used to study D_U via its associated graded ring.

Writing $\xi_i = \partial_i \mod F_0 D_U$ for the image of ∂_i in $\mathrm{gr}^F D_U$, we see that

$$F_\ell D_U / F_{\ell-1} D_U = \bigoplus_{|\alpha|=\ell} \mathcal{O}_U \xi^\alpha,$$

and hence

$$\mathrm{gr}^F D_U = \mathcal{O}_U[\xi_1, \dots \xi_n] = \mathbb{C}[x_1, \dots, x_n, \xi_i, \dots, \xi_n]$$

is simply the polynomial ring in $2n$ variables.

For an order ℓ differential operator $P \in F_\ell D_U$ we define its *principal symbol* as its image in $\mathbb{C}[x_1, \dots, x_n, \xi_i, \dots, \xi_n]$. Intuitively the principal symbol of a differential operator P is a polynomial representing P obtained by taking the highest order term of P and replacing each partial derivative ∂_i by a variable ξ_i.

The cotangent bundle. The construction globalises to give a sheaf of graded $\mathcal{O}_X$-algebras $\mathrm{gr}^F D_X$, whose zeroth graded piece is simply $\mathcal{O}_X$. The degree 1 principal symbols (corresponding to locally defined differential operators of order 1) can be identified with vector fields on X. In other words, the first graded piece is the tangent sheaf TX. Since the algebra on each patch is freely generated by its degree 1 part, we have a canonical identification

$$\mathrm{gr}^F D_X \cong \mathrm{Sym}\, TX = \mathcal{O}_{T^\vee X}$$

of $\mathrm{gr}^F D_X$ with the structure sheaf of the cotangent bundle $T^\vee X$. Thus the principal symbols ξ_i correspond to local coordinates dx_i on the cotangent bundle.

3.2 Deformation quantisation

We define the notion of a *filtered deformation* of a graded algebra, and realise the example above as a special case. To avoid working with sheaves, we work with an smooth affine complex variety X. The local constructions can be easily shown to glue, and the results generalise to sheaves of algebras on smooth complex manifolds.

Definition 29. A *filtered deformation* of a graded algebra B is a filtered algebra A, such that the associated graded ring of A satisfies $\mathrm{gr}A \cong B$ (as graded algebras). If B is commutative and A is noncommutative, A is said to be a *filtered quantisation* or *deformation quantisation* of B.

Thus the noncommutative algebra of differential operators D_X on an algebraic variety X is then a deformation quantisation of the commutative algebra $\mathcal{O}(T^\vee X) \cong \mathrm{gr}^F D_X$ of functions on the cotangent bundle of X.

The Rees algebra. Given a commutative $\mathbb{C}$-algebra B, we can view a deformation quantisation of B as an associative $A_\hbar$, which is flat as a $\mathbb{C}[\![\hbar]\!]$-module, together with an isomorphism $A_\hbar/(\hbar) \cong B$. The algebra $A_\hbar$ is constructed as the *Rees algebra* of A.

Definition 30. The Rees algebra of a filtered $\mathbb{C}$-algebra A is the $\mathbb{C}[\![\hbar]\!]$-algebra

$$A_\hbar = \bigoplus_{l=0}^{\infty} (F_l A) \hbar^l \subseteq A[\![\hbar]\!].$$

It follows from this definition that $A_\hbar/(\hbar) \cong \mathrm{gr}^F A$. Since A can be recovered from the Rees algebra by looking at its graded components, the two viewpoints are equivalent.

In the context of differential operators, write $\overline{D_X}$ for the Rees algebra of $A = D_X$. Then from the above discussion, this is a deformation quantisation of $\mathrm{gr}^F D_X \cong \mathcal{O}(T^\vee X)$. More concretely, if D_X is generated by the variables $\{x_i, \partial_i\}$, with $x_i \in F_0 D_X$, $\partial_i \in F_1 D_X$, then the Rees ring $\overline{D_X}$ is generated by $\{x_i, \hbar\partial_i\}$. The commutation relations $[x_i, \partial_j] = \delta_{ij} \in F_1 D_X$ become $[x_i, \hbar\partial_j] = \hbar\delta_{ij} \in F_1 D_X \hbar$. Setting $\xi_i := \hbar\partial_i$, the Rees ring is then the algebra

$$\overline{D_X} = \frac{\mathbb{C}[x_1, \ldots, x_n, \xi_1 \ldots, \xi_n]}{([\xi_i, x_j] - \hbar\delta_{ij})}.$$

In the limit $\hbar \to 0$, this becomes the commutative algebra $\mathbb{C}[x_1, ..., x_n, \xi_1, ..., \xi_n]$.

The induced Poisson structure. Deforming a commutative algebra often endows it with additional structure, namely a *Poisson bracket*. Since the cotangent bundle of a smooth complex manifold has a natural deformation quantisation given by the algebra of differential operators, this gives a natural symplectic structure giving rise to rich geometry.

Recall that a *Poisson algebra* is a commutative, associative unital algebra $(B, *)$ equipped with a *Poisson bracket* $\{\cdot, \cdot\} : B \times B \to B$ i.e. a Lie bracket which satisfies the Leibnitz identity

$$\forall a, b, c \in A, \quad \{a, b * c\} = \{a, b\} * c + b * \{a, c\}.$$

Thus for any $a \in B$, the map $\{a, \cdot\} : A \to A$ is a derivation. Such a bracket automatically exists whenever we have a filtered deformation of A.

Proposition 31. *If A is a noncommutative filtered algebra such that the associated graded ring $B \cong gr^F A$ is commutative, then B is canonically a Poisson algebra.*

Proof. Given $\tilde{a} \in F_m A/F_{m-1} A$ and $\tilde{b} \in F_n A/F_{n-1} A$, we want to construct an element $\{\tilde{a}, \tilde{b}\} \in \mathrm{gr}^F_A$ such that the axioms of a Poisson bracket are satisfied. Consider lifts $a, b \in A$ of the elements $\tilde{a}, \tilde{b}$, and recall from Proposition 28 that the commutator $ab - ba$ lies in $F_{m+n-1} A$. We define the Poisson bracket as

$$\{\tilde{a}, \tilde{b}\} = ab - ba \pmod{F_{m+n-2} A} \in F_{m+n-1} A/F_{m+n-2} A.$$

We can check that this construction is independent of the choice of a and b. Since A is associative, and we have defined the bracket via the commutator in A, it satisfies the axioms of a Poisson bracket. □

If X is a smooth complex algebraic variety, then the Poisson structure on the cotangent bundle defined locally as above is compatible with gluing and we get a Poisson bracket on $\mathcal{O}_{T^\vee X}$. Thus the deformation quantisation D_X of $\mathcal{O}(T^\vee X)$ naturally makes $T^\vee X$ a *Poisson manifold*. It can be checked that this Poisson bracket induces the canonical symplectic form on $T^\vee X$, given by $\omega = \sum d\xi_i \wedge dx_i$ in local coordinates defined above.

3.3 An introduction to $\mathcal{D}$-modules

Let X be a smooth variety over $\mathbb{C}$.

Definition 32. A D_X-*module* or simply a D-*module* M on X is a sheaf of $\mathcal{O}_X$-modules such that on each affine neighbourhood $U \subset X$, $M(U)$ is a D_U-module in the usual sense and this structure is compatible across affine neighbourhoods. Equivalently, a D-module is a sheaf M of $\mathcal{O}_X$-modules with a left action of D_X given by a map of $\mathcal{O}_X$-algebras $D_X \to \mathrm{End}_{\mathbb{C}}(M)$.

Example 33. The structure sheaf $\mathcal{O}_X$ is naturally a D-module, where the action is given by applying the differential operators to functions in $\mathcal{O}_X$. This extends to free $\mathcal{O}_X$-modules.

Remark 34. D_X can be seen as the algebra generated by vector fields on X, and the commutator coincides with the classical Lie bracket on vector fields. Given an element $v \in D_X$, it is customary to write the induced endomorphism of a D-module M as $\nabla_v : M \to M$. Then these maps satisfy

$$\nabla_{fv}(m) = f \cdot \nabla_v(m),$$
$$\nabla_v(f \cdot m) = v(f) \cdot m + f \cdot \nabla_v(m), \quad \text{and}$$
$$\nabla_{[v,w]}(m) = [\nabla_v, \nabla_w](m)$$

for all local sections $v, w \in D_X$, $f \in \mathcal{O}_X$, $m \in M$. In particular, if M is a vector bundle (i.e. a locally free sheaf of $\mathcal{O}_X$-modules) then a D-module structure on M is precisely a flat connection.

Good filtrations. For simplicity we will assume X is affine. Let M be a D_X-module and let F be the order filtration of D_X. A *compatible filtration* of M is a filtration by finitely generated $\mathcal{O}_X$-modules

$$0 = F_{-1}M \subset F_0M \subset F_1M \subset F_2M \ldots, \quad \bigcup_{\ell \geqslant 0} F_\ell M = M,$$

such that $(F_m D_X)(F_n M) \subseteq F_{m+n}M$. Then the *associated graded module* $\operatorname{gr}^F M := \bigoplus_{l \geqslant 0} F_\ell M / F_{\ell-1} M$ is naturally a graded module over $\operatorname{gr}^F D_X \cong \mathcal{O}_{T^\vee X}$.

We call the filtration F a *good filtration* if $\operatorname{gr}^F M$ is finitely generated over $\operatorname{gr}^F D_X$.

The above discussion can be extended to the non-affine case by considering filtrations on affine patches compatible with the sheaf structure. Recall that call a sheaf of modules over a sheaf of algebras is *coherent* if it is locally finitely generated. Then a filtration F on a D-module M is a *good filtration* if $\operatorname{gr}^F M$ is coherent over the sheaf of associated graded rings $\operatorname{gr}^F D_X$.

If M is coherent as a sheaf of D_X-modules, then the submodules $F_\ell D_X \cdot M$ are locally finitely generated and we have a canonical good filtration. In fact, existence of good filtrations is equivalent to coherence.

Theorem 35. *Any coherent D_X-module admits a good filtration, and conversely any D_X-module which admits a good filtration is coherent.*

Characteristic variety. Construction of the sheaf of commutative rings $\operatorname{gr}^F D_X$ from the sheaf of noncommutative rings D_X allows us to define a geometric invariant of a D_X-module M – the *characteristic variety*.

Recall that on an algebraic variety X, the *support* of a sheaf of $\mathcal{O}_X$-modules M is defined as the subvariety of points $\mathfrak{p}$ where the stalk $M_\mathfrak{p}$ is non-zero. On an affine neighbourhood $\operatorname{Spec} R \subset X$, this is the closed subvariety defined the annihilator of the R-module corresponding to M.

Now suppose X is a smooth complex variety with a coherent D_X-module M. By Theorem 35, M has a good filtration F so that $\operatorname{gr}^F M$ is a sheaf of $\operatorname{gr}^F D_X \cong \mathcal{O}_{T^\vee X}$-modules. This allows us to naturally think of $\operatorname{gr}^F M$ as a sheaf of modules on $\underline{\operatorname{Spec}}\, \mathcal{O}_{T^\vee X} \cong T^\vee X$ where $\underline{\operatorname{Spec}}$ denotes the global Spec construction.

Definition 36. The *characteristic variety* (or *singular support*) of M is the support of the coherent $\mathcal{O}_{T^\vee X}$-module $\operatorname{gr}^F M$, i.e.

$$\operatorname{Ch}(M) = \operatorname{Supp} \operatorname{gr}^F M \quad \subseteq T^\vee X.$$

Remark 37. There is a natural action of the multiplicative group $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ on the fibres of $T^\vee X$, and we call a subset $V \subseteq T^\vee X$ *conic* if it is stable under this action. It follows from the construction of the characteristic variety that $\operatorname{Ch}(M)$ is a conic subvariety of $T^\vee X$.

Example 38. Let $X = \mathbb{A}^1_x = \operatorname{Spec} \mathbb{C}[x]$, so that $D_X = \mathbb{C}[x, \partial]/([x, \partial] - 1)$. If F is the order filtration on D_X, we have $\operatorname{gr}^F D_X = \mathbb{C}[x, \xi]$ where $\xi = \partial \mod F_0 D_X$. Note that the total space of the cotangent bundle can then be identified with the affine space $\mathbb{A}^2_{(x,\xi)}$, with vector space structure coming from projection onto the first coordinate. Then all conic subvarieties are of the form $S \times \{0\} \cup T \times \mathbb{A}^1_\xi \subset \mathbb{A}^2_{(x,\xi)}$ for $S, T \subset \mathbb{A}^1_x$.

(i) Let $M = 0$ be the trivial D-module. Then $\operatorname{gr}^F M$ is 0, so that $\operatorname{Ann}_{\operatorname{gr}^F D_X} \operatorname{gr}^F M = \operatorname{gr}^F D_X$. Thus the characteristic variety $\operatorname{Ch}(M) = \mathbb{V}(\operatorname{gr}^F D_X) = \varnothing$ is the empty space.

(ii) Consider $M = D_X$ seen as a D-module over itself with the action given by left multiplication. Then the order filtration F of D_X is trivially a good filtration on the module M, and we consider the associated graded ring $\operatorname{gr}^F D_X = \operatorname{gr}^F M$ as a graded module over itself. The annihilator $\operatorname{Ann}_{\operatorname{gr}^F D_X} \operatorname{gr}^F M$ is the 0 ideal, so that the characteristic variety is given by the whole cotangent bundle $\operatorname{Ch} M \cong T^\vee X$.

(iii) Let $M = \mathcal{O}_X = \mathbb{C}[x]$ be the complex polynomial ring in one variable seen as a D-module under the natural action of $D_X = \mathbb{C}[x, \partial]/([x, \partial] - 1)$. Thus x acts by left multiplication and ∂ by differentiation with respect to the variable x. The filtration

$$F_\ell M = \begin{cases} 0 & \ell = -1 \\ M & \ell \geqslant 0 \end{cases}$$

is a good filtration of M, and the associated graded module $\operatorname{gr}^F M$ is isomorphic to M concentrated in degree 0. The action of $\operatorname{gr}^F D_X = \mathbb{C}[x, \xi]$ is such that x acts by multiplication, but since $\partial \in F_1 D_X$ we have for any $m \in M$,

$$\xi m = [\partial m] \in F_1 M / F_0 M = 0$$

i.e. ξ acts by 0 on $\mathrm{gr}^F M$. Thus $\mathrm{Ann}_{\mathrm{gr}^F D_X} \mathrm{gr}^F M = (\xi)$, and the characteristic variety $\mathrm{Ch}\,(M) = \mathbb{V}(\xi) \cong X$ is the zero-section of $T^\vee X$.

(iv) More generally, consider $M = D_X/D_X(\partial - \lambda)$ for some $\lambda \in \mathbb{C}$. This D-module can be thought of as being associated to the partial differential equation $(\partial - \lambda)f = 0$. We construct the associated graded module using the order filtration on D_X, i.e. $F_\ell M = F_\ell D_X \cdot u$ for $\ell \geqslant -1$. Then $\partial - \lambda \in D_X$ acts by 0 on M, whereas in the associated graded ring $\mathrm{gr}^F D_X$, the class $[\partial - \lambda]$ is equal to ξ. Thus the characteristic variety $\mathrm{Ch}\,(M)$ is again the zero-section $X \subset T^\vee X$. Note that we get the same characteristic variety for all $\lambda \in \mathbb{C}$.

(v) Let M be the D-module $M = D_X/D_X(x - a)$ for some $a \in \mathbb{C}$, generated by $u = 1 + D_X(x - a)$. The associated partial differential equation is $f(x - a) = 0$. Consider as above the good filtration constructed using the order filtration of D_X. The associated graded module $\mathrm{gr}^F M$ is generated by the class of u in degree 0, and hence the annihilator of $\mathrm{gr}^F M$ is precisely the ideal generated by $x - a \in \mathbb{C}[x, \xi]$. Thus the characteristic variety $\mathrm{Ch}\,(M) = \mathbb{V}(x - a) = \{a\} \times \mathbb{A}^1_\xi$ is the $x = a$ fibre inside $T^\vee X$. This module corresponds to a 'delta function distribution' supported at a.

The following result shows that the characteristic variety is truly a geometric invariant of a coherent D_X-module.

Theorem 39. *The characteristic variety of a coherent D_X-module constructed as above is independent of the choice of good filtration on it.*

Bernstein inequality. Recall that the cotangent bundle of any manifold has a canonical symplectic form. If X is a smooth complex variety with local coordinates as defined above, this form is given by $\omega = \sum d\xi_i \wedge dx_i$ and coincides with the Poisson bracket on $\mathcal{O}_{T^\vee X}$. The characteristic variety of any coherent D_X-module is well-behaved with respect to this symplectic structure, allowing us to estimate its dimension.

We recall a few notions from symplectic geometry. Let V be a vector space with a symplectic form $\omega : V \otimes V \to \mathbb{C}$. For any subspace $W \subset V$, define

$$W^\perp = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}$$

and say W is *involutive* if $W^\perp \subseteq W$, and *Lagrangian* if $W^\perp = W$. Note that the dimension of an involutive subspace $W \subset V$ satisfies

$$\frac{1}{2} \dim V \leqslant \dim W \leqslant \dim V,$$

and the lower bound is attained if and only if W is Lagrangian.

If X is a manifold with a symplectic form ω, we say a submanifold $Y \subset X$ is *involutive* (*Lagrangian*) if the tangent space $T_p Y$ is an involutive (Lagrangian) subspace of $T_p X$ for all points $p \in Y$. Corresponding constraints on $\dim Y = \dim T_p Y$ hold.

Then we have the following result for a smooth complex variety X.

Theorem 40 (Sato, Kawai, Kashiwara). *For any coherent D_X-module $M \neq 0$, the characteristic variety $Ch\,(M)$ is an involutive subvariety of $T^\vee X$ with respect to the canonical symplectic structure.*

Corollary 41 (Bernstein inequality). *If M is a non-zero coherent D_X-module, then we have*

$$\dim X \leqslant \dim \mathrm{Ch}\,(M) \leqslant 2 \dim X.$$

If M is a D-module for which the lower bound in Bernstein inequality is attained, we say M is *holonomic*. This is equivalent to $\mathrm{Ch}\,(M)$ being a Lagrangian subvariety of $T^\vee X$, and we will see that holonomic D-modules are well behaved in a number of ways.

3.4 D-modules and differential equations

The association between D-modules and differential equations alluded to in Example 38 is made precise by observing that Hom-spaces between D-modules naturally correspond to solution spaces of systems of differential equations.

To see this correspondence, consider an open subset $X \subseteq \mathbb{C}^n$ and let $\mathcal{O}_X$ be a ring of functions on X (algebraic/analytic/holomorphic) within which we wish to solve a system of differential equations. Let D_X be, as before, be the noncommutative ring of differential operators with coefficients in $\mathcal{O}_X$. Then $\mathcal{O}_X$ is naturally a left D_X-module, where any $f \in \mathcal{O}_X \subset D_X$ acts by left multiplication and the variables $\partial_i \in D_X$ act by partial differentiation.

A differential equation is then determined by a differential operator $P \in D_X$, and the solution set we seek is $\{f \in \mathcal{O}_X \mid P \cdot f = 0\}$. This can be phrased in terms of D-modules as follows.

Proposition 42. *For the left* D*-module* $D_X/D_X \cdot P$*, we have the natural isomorphism of additive groups*

$$\mathrm{Hom}_{D_X}(D_X/D_X \cdot P, \mathcal{O}_X) \cong \{f \in \mathcal{O}_X \mid P \cdot f = 0\}$$

Proof. Notice that D_X-module homomorphisms $D_X/D_X \cdot P \to \mathcal{O}_X$ are in one-to-one correspondence with D_X-module homomorphisms $D_X \to \mathcal{O}_X$ which map $P \mapsto 0$. Moreover any D_X-module homomorphism $\psi : D_X \to \mathcal{O}_X$ is determined uniquely by the image $\psi(1) \in \mathcal{O}_X$, and we have $\psi(P) = P \cdot \psi(1)$. Thus we have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{D_X}(D_X/D_X \cdot P, \mathcal{O}_X) &\cong \{\psi : D_X \to \mathcal{O}_X \mid P \cdot \psi(1) = 0\} \\ &\cong \{f \in \mathcal{O}_X \mid P \cdot f = 0\}\end{aligned}$$

as required. □

A similar description can be given for systems of multiple differential equations: note that in the above case of a single differential equation $Pf = 0$, the solution space was isomorphic to the space $\mathrm{Hom}_{D_X}(M, \mathcal{O}_X)$ where the module $M = D_X/D_X P$ is presented as the cokernel of the morphism $P : D_X \to D_X$. More generally, a system of k differential equations in ℓ unknown functions $(f_1, ..., f_\ell)$ can be written as

$$\sum_{j=1}^{\ell} P_{ij} f_j = 0, \quad 1 \leqslant i \leqslant k$$

for some differential operators $P_{ij} \in D_X$. Then the $\ell \times k$ matrix with entries P_{ij} gives a map $(P_{ij}) : D_X^k \to D_X^\ell$, and a proof similar to above shows the solution space to the system of equations is isomorphic to $\mathrm{Hom}_{D_X}(M, \mathcal{O}_X)$ where M is the cokernel of (P_{ij}).

Holonomic D**-modules.** A well known fact from complex analysis is that the space of holomorphic solutions to an ordinary differential equation in one variable is finite dimensional. This however is not always the case for partial differential equations on higher dimensional spaces X. We could attempt to classify systems of differential equations which do admit a finite dimensional space of holomorphic solutions, in light of the correspondence between D-modules and differential equations this is translated to a classification problem on D-modules. As we shall see, it is possible to place constraints on the D-module M which ensure finite dimensionality of the solution space $\mathrm{Hom}_{D_X}(M, \mathcal{O}_X)$.

Recall that a non-zero coherent D-module M on $X \subset \mathbb{C}^n$ is holonomic if its characteristic variety has minimal possible dimension (equal to $\dim X$ by the Bernstein inequality). We build some intuition to see why constraining the dimension of the characteristic variety is appropriate to control the dimension of the Hom space.

Consider a system of k differential equations $P_i f = 0$, $P_i \in D_X$, $i \in \{1, \dots, k\}$, in a single unknown function $f \in \mathcal{O}_X$. The D-module corresponding to this system sits in an exact sequence

$$D_X^k \xrightarrow{P} D_X \longrightarrow M \longrightarrow 0$$

where $P : D_X^k \to D_X$ is the map $(Q_1, \dots, Q_2) \mapsto \sum Q_i P_i$. Thus $M = D_X/I$, where $I = D_X P_1 + \cdots + D_X P_k$. Recall that the characteristic variety of M is the vanishing set of the annihilator of the associated graded module $\mathrm{gr}^F M$, where in light of Theorem 39 we may take F to be the order filtration. Since any differential operator $Q \in D_X$ gets mapped to its principal symbol $\sigma(Q) \in \mathrm{gr}^F D_X$ under the natural map, the annihilator of $\mathrm{gr}^F M$ in $\mathrm{gr}^F D_X$ will consist of the principal symbols $\sigma(Q)$ of differential operators Q belonging to the ideal I. It follows that the characteristic variety is $\mathrm{Ch}\,(M) = \bigcap_{Q \in I} \mathbb{V}(\sigma(Q)) \subset T^\vee X$.

Now intuitively, the more constrained a system is, the smaller its solution space. So the more equations we impose, i.e. the larger the ideal I, the better chance we have at obtaining a finite dimensional solution space. But the larger is I, the more principal symbols we have in the annihilator of $\mathrm{gr}^F M$, so the smaller is their vanishing set $\mathrm{Ch}\,(M)$. Therefore it would make sense to require the dimension of $\mathrm{Ch}(M)$ to be small for the space of solutions to the system corresponding to M to be finite dimensional. This is indeed true.

Theorem 43. *Let* M, N *be holonomic left* D_X*-modules. Then the space* $\mathrm{Hom}_{D_X}(M, N)$ *is finite dimensional.*

We have already seen in Example 38 that $\mathcal{O}_X$ is holonomic (its characteristic variety is the zero-section X). Thus in particular, when M is a holonomic D-module corresponding to a system of differential equations P, the solution solution space of P is finite dimensional. It is for this reason that holonomic modules are often called *maximally overdetermined.*

4. Non-commutative deformations of a point

Given a k-scheme X, the results of Section 1.3 suggest that the local properties of X can be inferred from the deformation theory of its skyscraper sheaves $\mathcal{O}_{X,x}$. In this section, we work with the more general situation of a (left) module M over an associative k-algebra A.

If M is one dimensional over k, then it is determined by a maximal left ideal $p \subset A$ corresponding to the kernel of the map $A \to \mathrm{End}_A(M)$. If A is commutative (so that M corresponds to the skyscraper sheaf of a point) then Proposition 6 shows the the functor $\mathcal{D}ef_M : \mathrm{cArt}_k^1 \to \mathrm{Set}$ is prorepresented by the completion $\widehat{A}_p$. Of course when A is not commutative, the best we can hope to recover from the classical deformation problem is an abelianisation of $\widehat{A}_p$. This indicates that we ought to consider instead the *non-commutative deformation theory* of M, where the test objects lie in Art_k^1, the category of all Artinian local k-algebras with residue field k.

Definition 44. Let M be any A-module. For an Artinian k-algebra $R \in \mathrm{Art}_k^1$, we define an R-*deformation of* M to be a left $A \otimes R^{op}$-module M_R that is free over R, with an isomorphism of A-modules $M_R \otimes_R k \to M$. Two such deformations M_R, M'_R are equivalent if there is an isomorphism $M_R \to M'_R$ of $A \otimes R$-modules compatible with the identifications

$$M_R \otimes_R k \to M \leftarrow M'_R \otimes_R k.$$

Write $\mathcal{D}ef_M$ for the resulting deformation functor $\mathrm{Art}_k^1 \to \mathrm{Set}$, which maps R to the set of R-deformations of M up to equivalence. Restricted to the full subcategory $\mathrm{cArt}_k^1 \subset \mathrm{Art}_k^1$, this is the classical deformation functor described in Section 1.1.

Deforming via the structure map. We can rephrase the problem in terms of deformations of the natural k-algebra map $\mu : A \to \mathrm{End}_k(M)$ as follows. An R-deformation M_R of M it is free over R hence is completely determined by its A-module structure, given by a map

$$\mu_R : A \to \mathrm{End}_k(M) \otimes R \quad (\cong \mathrm{End}_R(M \otimes R))$$

such that μ_R reduces modulo $\mathfrak{m}_R$ to the map μ. Two such maps $\mu_R, \mu'_R : A \to \mathrm{End}_k(M) \otimes R$ are equivalent if they differ by an inner k-algebra automorphism of $\mathrm{End}_k(M) \otimes R$.

Proposition 45. *If* M *is a one-dimensional* A*-module, then for* $R \in \mathrm{Art}_k^1$ *there are natural isomorphisms*

$$\begin{aligned}\mathcal{D}ef_M(R) &= \mathrm{Hom}_{\mathrm{Alg}_k^1}(A, R)/\{\textit{Inner automorphisms of } R\}\\ &= \mathrm{Hom}_{\mathrm{Alg}_k^1}(\widehat{A}_p, R)/\{\textit{Inner automorphisms of } R\}\end{aligned}$$

where $\widehat{A}_p \overset{\lim}{=\!\!\!\rightarrow} A/p^n$ *is the completion of* A *at* $p = \ker \mu$.

Proof. Since M is one dimensional, we have $\mathrm{End}_k(M) \otimes R \cong R$ and so deformations of μ are given by maps $\mu_R : A \to R$ that reduce modulo $\mathfrak{m}_R$ to μ, up to inner automorphisms of R. But considering μ to be an augmentation of A, this is equivalent to saying μ_R is a map of augmented k-algebras. Thus we have

$$\mathcal{D}ef_M(R) = \mathrm{Hom}_{\mathrm{Alg}_k^1}(A, R)/\{\text{Inner automorphisms of } R\}.$$

For the second isomorphism, note that R is Artinian so any map $A \to R$ that respects the augmentation should factor through A/p^n for sufficiently large n. □

Remark 46. If we restrict to cArt_k^1, then R has no nontrivial inner automorphisms. Moreover, any map $A \to R$ must factor uniquely through the surjection $A \to A_{Ab}$ whose kernel is the ideal generated by commutativity relations in A. Thus the restriction of $\mathcal{D}ef_M$ to cArt_k^1 is prorepresented by $\widehat{A}_{Ab,p}$, the completion of A_{Ab} at the image of $p = \ker \mu$. If A is commutative, this gives another proof of Proposition 6.

4.1 Deformations by lifting resolutions

Following the treatment of [ELS17], we give an explicit description of the deformation functor $\mathcal{D}ef_M$ in the general case by choosing a a projective resolution

$$\cdots \to P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \to M.$$

Definition 47. For $R \in \mathrm{Art}_k^1$, an R-*lift* of the complex $(P_\bullet, d)$ is a complex $(P_\bullet \otimes R, \partial)$ of $A \otimes R^{op}$-modules that reduces modulo $\mathfrak{m}_R$ to $(P_\bullet, d)$. Two such liftings are equivalent if there is a chain isomorphism $(P_\bullet \otimes R, \partial) \to (P_\bullet \otimes R, \partial')$ inducing the identity map on $(P_\bullet, d)$.

We now show that isomorphism classes of such R-lifts are in bijection with the set $\mathcal{D}ef_M(R)$.

Lemma 48. *If* $(P_\bullet \otimes R, \partial)$ *is an* R*-lift of the complex* $(P_\bullet, d)$ *defined above, then* $(P_\bullet \otimes R, \partial)$ *is a projective resolution of the* $A \otimes R^{op}$*-module* $M_R = \operatorname{coker} \partial_0$ *which is naturally an* R*-deformation of* M.

Proof. It is clear that each $P_\bullet \otimes R$ is a projective $A \otimes R$-module. To show that $(P_\bullet \otimes R, \partial)$ is indeed a resolution of $\operatorname{coker} \partial_0$, we use the fact that the surjection $R \to k$ in Art_k^1 can be factored as

$$R = R_n \xrightarrow{u_n} R_{n-1} \to \ldots \xrightarrow{u_1} R_0 = k$$

where each u_i is a surjection with one-dimensional kernel (such maps are called *small surjections*). This gives us surjective chain maps

$$(P_\bullet \otimes R, \partial) = (P_\bullet \otimes R_n, \partial^n) \to (P_\bullet \otimes R_{n-1}, \partial^{n-1}) \to \cdots \to (P_\bullet \otimes R_0, \partial^0) = (P_\bullet, d)$$

where each map is reduction modulo $\ker u_i$. It is immediate that each $(P_\bullet \otimes R_i, \partial^i)$ is an R_i-lift of $(P_\bullet, d)$. Writing $K := \ker u_i$, note we have a short exact sequence of chain complexes

$$\begin{array}{ccccccccc}
\cdots & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_2 \otimes K & \longrightarrow & P_1 \otimes K & \longrightarrow & P_0 \otimes K & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_2 \otimes R_i & \longrightarrow & P_1 \otimes R_i & \longrightarrow & P_0 \otimes R_i & \longrightarrow & M_{R_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_2 \otimes R_{i-1} & \longrightarrow & P_1 \otimes R_{i-1} & \longrightarrow & P_0 \otimes R_{i-1} & \longrightarrow & M_{R_{i-1}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

where the maps $P_n \otimes K \to P_{n-1} \otimes K$ are the restrictions of ∂_n^i. Since K is one dimensional, it is generated by some $r \in R_i$. Then for any $p \otimes r \in P_n \otimes K$, we have

$$\begin{aligned}
\partial_n^i(p \otimes r) &= r \cdot \partial_n^i(p \otimes 1) \\
&= r \cdot (d_n p \otimes 1 + q) \\
&= d_n p \otimes r + r \cdot q
\end{aligned}$$

for some $q \in P_{n-1} \otimes \mathfrak{m}_{R_i}$. But K is a one-dimensional R_i-module, so must be annihilated by $\mathfrak{m}_{R_i}$. Thus $r \cdot q = 0$, and we have $\partial_n^i(p \otimes r) = d_n p \otimes r$. In other words, the complex $(P_\bullet \otimes K, \partial^i)$ is isomorphic to $(P_\bullet, d)$. Taking the long exact sequence on homology, we see that for all $j > 0$

$$H_j(P_\bullet \otimes R_n, \partial^n) \cong H_j(P_\bullet \otimes R_{n-1}, \partial^{n-1}) \cong \cdots \cong H_j(P_\bullet \otimes R_0, \partial^0) \cong 0$$

i.e. $(P_\bullet \otimes R, \partial)$ is a projective resolution of $M_R = \operatorname{coker} \partial_0$.

It is clear that M_R reduces modulo $\mathfrak{m}_R$ to M. An inductive argument using the short exact sequence above shows M_R is free over R, hence is an R-deformation of M. ☐

Lemma 49. *If* M_R *is an* R*-deformation of* M*, then the complex* $(P_\bullet, d)$ *has an* R*-lift* $(P_\bullet \otimes R, \partial)$ *such that* $\operatorname{coker} \partial_0 \cong M_R$.

Proof. It suffices work with small surjections. Suppose $R \to S$ is a small surjection with kernel K, and M_S is the reduction modulo K of M_R such that the S-lift $(P_\bullet \otimes S, \partial'_\bullet)$ satisfies $M_S \cong \operatorname{coker} \partial'_0$. It is clear that the sequence

$$0 \to M \otimes K \to M_R \to M_S \to 0$$

is exact. Then consider the diagram

$$
\begin{array}{ccccccccccc}
\cdots & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \dashrightarrow & P_2 \otimes K & \dashrightarrow & P_1 \otimes K & \dashrightarrow & P_0 \otimes K & \dashrightarrow & M \otimes K & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \dashrightarrow & P_2 \otimes R & \dashrightarrow & P_1 \otimes R & \dashrightarrow & P_0 \otimes R & \dashrightarrow & M_R & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_2 \otimes S & \longrightarrow & P_1 \otimes S & \longrightarrow & P_0 \otimes S & \xrightarrow{\rho_S} & M_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

with exact columns. Since P_0 is projective, we can lift ρ_S to a morphism $\rho_R : P_0 \otimes R \to M_R$. If K is generated by r, we see that for any $p \otimes r \in P_0 \otimes K$ we have $\rho_R(p \otimes r) = r \cdot \rho_R(p \otimes 1)$. But writing $\rho : P_0 \to M$ for the natural map, we have

$$
\begin{aligned}
\rho(p) &= \rho_S(p \otimes 1) \mod \mathfrak{m}_S \\
&= \rho_R(p \otimes 1) \mod \mathfrak{m}_R.
\end{aligned}
$$

Combined with the fact that $\mathfrak{m}_R K = 0$, this implies $\rho_R(p \otimes r) = \rho(p) \otimes 1$. In particular the map $P_0 \otimes K \to M \otimes K$ is surjective, so by the Snake lemma we have a surjection $\ker(\rho_R) \to \ker(\rho_S)$. This allows us to inductively lift the differentials ∂'_n, getting a lift $(P_\bullet \otimes R, \partial)$ as required. □

It is clear from the above constructions that equivalent deformations of M correspond to equivalent lifts of $(P_\bullet, d)$. Thus the deformation functor can be described as

$$\mathcal{D}ef_M(R) = \{R\text{-lifts of the complex } (P_\bullet, d) \text{ up to isomorphism}\}.$$

A hint of Maurer-Cartan. Consider the graded A-module $P := \bigoplus_i P_i$. All R-lifts of $(P_\bullet, d)$ have the underlying graded $A \otimes R^{op}$-module $P \otimes R$ so an R-lift is completely determined by its differential ∂, seen as a degree 1 endomorphism of $P \otimes R$ which satisfies $\partial \circ \partial = 0$ and reduces modulo $\mathfrak{m}_R$ to d.

Observe we have the isomorphism of graded modules

$$\mathrm{End}_{A \otimes R}(P \otimes R) \cong \mathrm{End}_A(P) \otimes R.$$

Using the splitting $R = k \oplus \mathfrak{m}_R$ we see that ∂ is determined by the degree 1 element $\delta := \partial - d \in \mathrm{End}_A(P) \otimes \mathfrak{m}_R$. The condition $\partial \circ \partial = 0$ is equivalent to

$$(1) \qquad (d \circ \delta + \delta \circ d) + \delta \circ \delta = 0.$$

Moreover, it is clear that any degree 1 element $\delta \in \mathrm{End}_A(P) \otimes \mathfrak{m}$ satisfying (1) determines an R-lift of $(P_\bullet, d)$.

Two such elements δ, δ' determine equivalent lifts if and only if there is a degree 0 element $\varphi \in \mathrm{End}_A(P) \otimes \mathfrak{m}_R$ such that the map $1 + \varphi \in \mathrm{End}_A(P) \otimes R$ is an isomorphism of chain complexes $(P_\bullet, d + \delta) \to (P_\bullet, d + \delta')$. Since R is Artinian, $1 + \varphi$ is already invertible as map of graded modules, and we require φ to satisfy

$$(2) \qquad \delta' = ((1 + \varphi) \circ \delta - (d \circ \varphi - \varphi \circ d)) \circ (1 + \varphi)^{-1}.$$

Any degree 0 element $\varphi \in \mathrm{End}_A(P) \otimes \mathfrak{m}_R$ satisfying (2) gives an equivalence of lifts between δ and δ'. Thus we have a description of the deformation functor

$$\mathcal{D}ef_M(R) = \frac{\{\text{degree 1 elements of } \mathrm{End}_A(P) \otimes \mathfrak{m}_R \text{ satisfying (1)}\}}{\{\text{equivalences given by (2)}\}}.$$

4.2 Deformation functors from differentially graded algebras

We provide a succinct reformulation of the results of Section 4.1 in terms of *differentially graded Lie algebras.*

Definition 50. A differentially graded Lie algebra (DGLA) Γ is given by a cochain complex $(\Gamma^\bullet, \underline{d})$ of k-vector spaces with a bilinear graded bracket $[\cdot,\cdot] : \Gamma^m \times \Gamma^n \to \Gamma^{m+n}$ satisfying the following compatibilities for homogeneous elements $g_i \in \Gamma^{m_i}$:

$$\underline{d}[g_1, g_2] = [\underline{d}g_1, g_2] + (-1)^{m_1}[g_1, \underline{d}g_2],$$
$$[g_1, g_2] + (-1)^{m_1 m_2}[g_2, g_1] = 0,$$
$$[g_1, [g_2, g_3] + (-1)^{m_3(m_1+m_2)}[g_3, [g_1, g_2]] + (-1)^{m_1(m_2+m_3)}[g_2, [g_3, g_1]] = 0.$$

We say Γ is *nilpotent* if its central series $\Gamma \supset [\Gamma, \Gamma] \supset [\Gamma, [\Gamma, \Gamma]] \supset \ldots$ stabilises to zero.

Γ has an associated *Maurer-Cartan equation*, the solutions of which form the set of *Maurer-Cartan elements*

$$MC(\Gamma) = \left\{ x \in \Gamma^1 \mid \underline{d}x + \frac{1}{2}[x, x] = 0 \right\}.$$

Deformations from Maurer-Cartan elements. For an A-module M, we construct a DGLA whose Maurer-Cartan elements naturally give deformations of M. Fix a projective resolution $(P_\bullet, d)$ of M with underlying graded module $P = \bigoplus_i (P_i)$, so that $\mathrm{End}_A(P)$ is a graded algebra with multiplication given by composition. Then the degree one map $\underline{d} : \mathrm{End}_A(P) \to \mathrm{End}_A(P)$ given on homogeneous elements by

$$\underline{d}\varphi = d \circ \varphi - (-1)^{\deg \varphi} \varphi \circ d$$

satisfies the graded Leibniz rule, giving $\mathrm{End}_A(P)$ it the structure of a differential graded algebra (DGA). To get a differential graded Lie algebra, we take the commutator bracket

$$[\varphi, \vartheta] = \varphi \circ \vartheta - (-1)^{\deg \varphi \cdot \deg \vartheta} \vartheta \circ \varphi.$$

The bracket and the differential are carried over to $\mathrm{End}_A(P) \otimes \mathfrak{m}_R$, giving us a DGLA whose Maurer-Cartan elements are precisely the solutions to (1).

Observe that the DGLA $\mathrm{End}_A(P) \otimes \mathfrak{m}_R$ is always nilpotent, giving the degree 0 component a well-defined group operation via the Baker-Campbell-Hausdorff formula

$$\varphi * \vartheta = \varphi + \vartheta + \frac{1}{2}[\varphi, \vartheta] + \frac{1}{12}[\varphi, [\varphi, \vartheta]] + \cdots.$$

This is the *gauge group* of the DGLA, written $G(\mathrm{End}_A(P) \otimes \mathfrak{m}_R)$. It naturally acts on the set $MC(\mathrm{End}_A(P) \otimes \mathfrak{m}_R)$ via a *gauge action* which is the exponential of the Lie algebra action. An explicit check shows that $\varphi \in G(\mathrm{End}_A(P) \otimes \mathfrak{m}_R$ identifies two Maurer-Cartan elements δ, δ' via this action if and only if the relation (2) holds, so we see that

$$\mathcal{D}ef_M(R) = \frac{MC(\mathrm{End}_A(P) \otimes \mathfrak{m}_R)}{G(\mathrm{End}_A(P) \otimes \mathfrak{m}_R)}.$$

Example 51. If $R = k[\epsilon]$ is the ring of dual numbers, then we see that the Maurer-Cartan equation becomes $\underline{d}\delta = 0$. Thus Maurer-Cartan elements of $\mathrm{End}_A(P) \otimes \mathfrak{m}_R$ are given by $\delta = f \otimes \epsilon$ where $f : P_\bullet \to P_\bullet$ satisfies $f \circ d = d \circ f$, i.e. f is a chain map.

Suppose the degree zero map $h : P_\bullet \to P_\bullet$ is such that the gauge action of $h \otimes \epsilon$ identifies $f \otimes \epsilon$ with $g \otimes \epsilon$. Then we have $g = f + d \circ h - h \circ d$ i.e. h is a chain homotopy between f and g. Thus the tangent space of $\mathcal{D}ef_M$ is given by the space of homotopy classes of degree 1 chain maps $P_\bullet \to P_\bullet$, i.e.

$$\mathcal{D}ef_M(k[\epsilon]) = \mathrm{Ext}^1_A(M, M),$$

Remark 52. Both $MC(-)$ and $G(-)$ can be seen as functors on the category of nilpotent DGLAs, and the action of the Gauge group on the solutions of the Maurer-Cartan equation respects this functoriality. This gives a way to associate a deformation functor to any differentially graded lie algebra $\Gamma_\bullet$, via

$$R \mapsto \frac{MC(\Gamma_\bullet \otimes \mathfrak{m}_R)}{G(\Gamma_\bullet \otimes \mathfrak{m}_R)}.$$

[Sze99] gives an introduction to Deligne's philosophy that every deformation problem (in characteristic zero) must arise from a DGLA in this fashion.

4.3 Deformation functors from A_∞-algebras

The construction of Section 4.2, while functorial, leaves us to deal with the infinite dimensional chain complex $\mathrm{End}_A(P)$ given in degree n by $\prod_{i\in\mathbb{Z}} \mathrm{Hom}_A(P_{i+n}, P_i)$. It is standard that the deformation functor only depends on the differentially graded (Lie) algebra up to quasi-isomorphism, so we can hope to replace $\mathrm{End}_A(P)$ by a complex which has finite dimensional components. The obvious candidate is the cohomology complex $\mathrm{Ext}_A(M, M)$ with the induced multiplication (i.e. the Yoneda product), but often there is no quasi-isomorphism of DGAs between the two– too much information is lost when passing to homology. However, the process of *homological perturbation* allows us to put additional structure on $\mathrm{Ext}_A(M, M)$ which does allow one to reconstruct the original DGA.

A brief introduction to A_∞algebras. The structure alluded to above is that of an A_∞ *algebra*, which is given by infinitely many multilinear maps on the complex which can be interpreted as 'higher homotopies' [see Seg08, Appendix A for the geometric motivation].

Definition 53. An A_∞ *algebra* is a $\mathbb{Z}$-graded vector space $V = \bigoplus_i V_i$ with a sequence of linear maps $m_n : V^{\otimes n} \to V$ for $n \geqslant 1$ such that m_n has degree $2-n$, and satisfies

$$\sum_{n=r+s+t} (-1)^{r+st} m_{r+1+t}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0. \tag{3}$$

Thus we have $m_1 \circ m_1 = 0$ i.e. (V, m_1) is a cochain complex, and $m_1 \circ m_2 = m_2(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$ i.e. m_2 is a product satisfying the Leibniz rule. The higher multiplications give homotopies up to which m_2 is associative.

A morphism of A_∞ algebras $f : V \to W$ is likewise given by a sequence of maps $f_n : V^{\otimes n} \to W$ for $n \geqslant 1$, such that f_n has degree $n-1$ and satisfies

$$\sum_{n=r+s+t} (-1)^{r+st} f_{r+t+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = \sum_{n=i_1+\ldots+i_r} (-1)^{\sum_j (r-j)(i_j-1)} m_r(f_{i_1} \otimes \ldots \otimes f_{i_r}). \tag{4}$$

Thus $f_1 \circ m_1 = m_1 \circ f_1$ i.e. f_1 is a chain map, and $f_1 \circ m_2 = m_2(f_1 \circ f_1) + m_1 f_2 + f_2(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$ i.e. f_1 comutes with m_2 up to a homotopy given by f_2. The higher components of f can be interpreted as homotopies up to which lower f_is are compatible with the multiplications.

We say f is a quasi-isomorphism if f_1 is a quasi-isomorphism.

Example 54. Every differentially graded algebra with underlying vector space V is naturally an A_∞ algebra by setting m_1 to be the differential, m_2 to be the product, and all higher multiplications to be 0. The cohomology complex $H(V)$ has a differential $m_1 = 0$ and inherits and m_2 from V, but the natural map $f_1 : H(V) \to V$ does not necessarily commute with the multiplication. For $a, b \in H(V)$, we however do have that $m_2(f_1(a), f_1(b)) - f_1(m_2(a, b))$ is the boundary of some $f_2(a, b)$, and this choice can be made for all a, b in a way that f_2 is bilinear. Thus we have $f_2 : H(V)^{\otimes 2} \to V$ satisfying

$$f_1(m_2(a, b)) = m_2(f_1(a), f_1(b)) + m_1 f_2(a, b).$$

This is precisely the relation (4) for $n = 2$. Now if we were to construct $f_3 : H(V)^{\otimes 3} \to V$ we would have to consider

$$f_2(m_2(a, b), c) + f_2(a, m_2(b, c)) + m_2(f_1(a), f_2(b, c)) + m_2(f_2(a, b), f_1(c)).$$

Let $m_3(a, b, c) \in H(V)$ be the cohomology class of this quantity, then it is equal to

$$m_1 f_3(a, b, c) + f_1 m_3(a, b, c)$$

for some $f_3(a, b, c) \in V$. Kadeishvili showed these and higher m_ns, f_ns can be inductively chosen to be multilinear maps on $H(V)$ that satisfy relations (3) and (4). This process is called *homological perturbation*.

Theorem 55 (Kadeishvili). *Let V be a differentially graded algebra. Then its cohomology $H(V)$ has an A_∞ structure with $m_1 = 0$, m_2 inherited from V such that there is a quasi-isomorphism $H(V) \to V$ of A_∞ algebras lifting the identity of $H(V)$. Moreover, this structure is unique up to isomorphism of A_∞ algebras.*

In fact, the converse is true as well: any A_∞ algebra with $m_1 = 0$ can be realised as the cohomology complex of some DGA. Thus dealing with A_∞-algebras is not too different from dealing with DGAs.

Constructing the deformation functor. We can associate a deformation functor to an A_∞-algebra (V, m_i) by considering zeroes of the *Homotopy Maurer-Cartan function*

$$\mathrm{HMC} : V_1 \to V_2$$
$$\mathrm{HMC}(a) = \sum_{n \geqslant 1} m_n(a^{\otimes n})$$

up to Gauge equivalence. Explicitly, we set

$$\mathcal{D}ef_{(V,m_i)}(R) = \{a \in V_1 \otimes \mathfrak{m}_R \mid \mathrm{HMC}(a) = 0\} \;/\; \sim$$

where the equivalence relation $\sim$ is generated by an action of $V_0 \otimes \mathfrak{m}_R$ on $V_1 \otimes \mathfrak{m}_R$. If V is a DGA (i.e. $m_n = 0$ for $n \geqslant 3$) then this definition coincides with that of Section 4.2 (where the Lie bracket is given by the commutator). Moreover, the functor only depends on the A_∞-algebra up to quasi-isomorphism.

We motivated the study of A_∞-deformations by suggesting that a complex with finite dimensional graded parts is easier to handle. To exhibit a concrete benefit of this approach we will now show that whenever (V, m_i) is an A_∞-algebra with $\dim V_1 < \infty$, the corresponding deformation functor is essentially prorepresentable by a power-series ring in $\dim V_1$-many variables. To begin, observe the A_∞-structure gives us a map

$$m = \bigoplus m_i : TV_1 \to V_2$$

where $TV_1 = \bigoplus_{n \geqslant 0} V_1^{\otimes n}$ is the tensor algebra on V_1. Noting that $(TV_1)^\vee \cong \widehat{T}(V_1^\vee) = \prod_{n \geqslant 0} (V_1^\vee)^{\otimes n}$ is the completed tensor algebra on $V_1^\vee$, we see that dualising m gives a map

$$m^\vee : V_2^\vee \to \widehat{T}(V_1^\vee).$$

Write $(V_2^\vee)$ for the ideal of $\widehat{T}(V_1^\vee)$ generated by the image of $m^\vee$. The algebra $\widehat{T}(V_1^\vee)$ is naturally augmented by the maximal ideal $\prod_{n \geqslant 1} (V_1^\vee)^{\otimes n}$, and quotienting by $(V_2^\vee)$ preserves this augmentation since the image of $\mathrm{HMC}^\vee$ has no degree 0 component. We have the following result.

Lemma 56. *If V is an A_∞-algebra such that V_1 is finite dimensional, then for any $R \in \mathrm{Art}^1_k$ there is a functorial isomorphism*

$$\mathrm{Hom}_{\mathrm{Alg}^1_k}\left(\frac{\widehat{T}(V_1^\vee)}{(V_2^\vee)}, R\right) = \{v \in V_1 \otimes \mathfrak{m}_R \mid \mathrm{HMC}(v) = 0\}.$$

Proof. Elements of $V_1 \otimes \mathfrak{m}_R \cong \mathrm{Hom}_k(V_1^\vee, \mathfrak{m}_R)$ correspond precisely to morphisms of k-algebras $\widehat{T}(V_1^\vee) \to R$ that respect the augmentation (here we use that V_1 is finite dimensional). Explicitly, $v \in V_1 \otimes \mathfrak{m}_R$ corresponds to the map

$$f_v : \xi \mapsto \xi\left(\sum_{i \geqslant 0} v^{\otimes i}\right)$$

where we use the identification $\widehat{T}(V_1^\vee) \otimes TV_1 \otimes R \cong R$, noting that the sum $\sum_{i \geqslant 0} v^{\otimes i} \in TV_1 \otimes R$ is finite because $\mathfrak{m}_R$ is nilpotent. Now for a generator $\xi = m^\vee(w)$ of the ideal $(V_2^\vee)$, we have

$$f_v(m^\vee(w)) = w \circ m\left(\sum_{i \geqslant 0} v^{\otimes i}\right) = w(\mathrm{HMC}(v))$$

so $f_v : \widehat{T}(V_1^\vee)$ vanishes on the ideal $(V_2^\vee)$ if and only if $\mathrm{HMC}(v) = 0$. Thus we have the required isomorphism

$$\mathrm{Hom}_{\mathrm{Alg}^1_k}\left(\frac{\widehat{T}(V_1^\vee)}{(V_2^\vee)}, R\right) = \{v \in V_1 \otimes \mathfrak{m}_R \mid \mathrm{HMC}(v) = 0\}.$$

Functoriality follows from the fact that all isomorphisms involved were natural. □

The A_∞-deformation theory of a point. Going back to the deformation problem of a module M over an associative algebra A, we see that the complex $\mathrm{Ext}_A(M, M)$ has an A_∞-structure that makes it quasi-isomorphic to the differentially graded endomorphism algebra $\mathrm{End}_A(P_\bullet, P_\bullet)$ where $P_\bullet \to M$ is a projective resolution. Moreover, this A_∞-algebra controls the deformation functor $\mathcal{D}ef_M$ in the sense described above.

The statement is cleanest when M is simple. To use Lemma 56, we need to place an additional constraint on M, namely we require that $\mathrm{Ext}^1_A(M, M)$ is finite dimensional (this holds automatically when A is Noetherian).

Theorem 57 (Segal). *If M is a simple A-module such that $\mathrm{Ext}^1_A(M,M)$ finite dimensional, then the functor $\mathcal{D}ef_M : \mathrm{Art}^1_k \to$ Set is given by*

$$\mathcal{D}ef_M(R) = \mathrm{Hom}_{\mathrm{Alg}^1_k}\left(\frac{\widehat{T}(\mathrm{Ext}^1_A(M,M)^\vee)}{(\mathrm{Ext}^2_A(M,M))^\vee}, R\right) \,/\, \{\textit{Inner automorphisms of } R\}$$

Proof. We know that $\mathcal{D}ef_M = \mathcal{D}ef_{\mathrm{Ext}_A(M,M)}$ is given by zeroes of the homotopy Maurer-Cartan function up to Gauge equivalence. From Lemma 56 we have that the zeros of the homotopy Maurer-Cartan function for $\mathrm{Ext}_A(M,M) \otimes R$ are precisely given by

$$\mathrm{Hom}_{\mathrm{Alg}^1_k}\left(\frac{\widehat{T}(\mathrm{Ext}^1_A(M,M)^\vee)}{(\mathrm{Ext}^2_A(M,M))^\vee}, R\right)$$

so it suffices to show that the gauge action is precisely by inner automorphisms of R.

Since M is simple, we have $\mathrm{Ext}^0_A(M,M) \cong k$ so that the Gauge group has underlying set $\mathfrak{m}_R$. The element $r \in \mathfrak{m}_R$ acts on $f : \frac{\widehat{T}(\mathrm{Ext}^1_A(M,M)^\vee)}{(\mathrm{Ext}^2_A(M,M))^\vee} \to R$ by mapping it to $r * f : a \mapsto (1+r)^{-1}a(1+r)$. Since conjugation by $u + r$ for $u \in R^\times$ a unit is the same as conjugation by $1 + u^{-1}r$, it follows that the gauge action identifies morphisms that differ by an inner automorphism of R. ☐

When M is one-dimensional, i.e. given by a surjection $\mu : A \to k$ with kernel p, we can use this to obtain a presentation of the completion $\widehat{A}_p$.

Corollary 58. *Let $\widehat{A}_p$ be the completion of A at a maximal ideal p. Writing $M = A/p$, if $\mathrm{Ext}^1_A(M,M)$ is finite dimensional then we have the isomorphism*

$$\widehat{A}_p = \frac{\widehat{T}(\mathrm{Ext}^1_A(M,M)^\vee)}{(\mathrm{Ext}^2_A(M,M)^\vee)}.$$

Proof (sketch). Forgetting the action of inner automorphisms, we saw in Proposition 45 that the functor $\mathcal{D}ef_M$ is essentially prorepresented by $\widehat{A}_p$. Thus by uniqueness of the prorepresenting object (i.e. Yoneda's lemma), we expect the result to hold and all that remains to be checked is that the uniqueness argument holds with little change even after we quotient by inner automorphisms. ☐

Remark 59. In [Seg08], Ed Segal proves the results of this section in the slightly more general setting of *n-pointed Algebras*, which are algebras equipped with augmentation maps to k^n. There is a natural test category of n-pointed Artinian (semilocal) rings, and the deformation problem can be interpreted as deforming a *set of points*. It is here that the noncommutativity of test objects shines− the prorepresenting algebra can be realised as a path algebra on an n-pointed quiver, but passing to the abelianisation kills all paths between distinct vertices thus decomposing the deformation problem for n points into n distinct deformation problems for single points.

The proofs are essentially identical to the one-pointed case presented here, except with more notational baggage to keep track of the additional structure.

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