Riemann surfaces

Based on a lecture series by Ruadhai Dervan.

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Lecture 1, 10/10/23

This course aims to be an introduction to algebraic geometry, complex geometry, and parts of geometric analysis. We will explain the theory of Riemann surfaces, these are smooth 2-manifolds with a lot of structure which enables the notion of a holomorphic function (in one variable.) We will use this setting to explore various facets of modern geometry– the study of differential geometry (differential forms and Hodge structures), of algebraic geometry (compact Riemann surfaces are naturally projective varieties), geometric analysis, and to a lesser extent topology and complex analysis.

§ 0.1 About. This course on Riemann surfaces is a part of the SMSTC training program for graduate students. The lectures were delivered in-person in the University of Glasgow, and were transcribed live. The notes can be found online at `https://pas201.user.srcf.net/documents/2023-riemann-surfaces.pdf`. Errors and corrections should be communicated to by email to `parth.shimpi@glasgow.ac.uk`.

§ 0.2 Assessment. By presentation on some advanced topics after the lectures. There will also be two exercise sheets, but these won't be marked.

§ 0.3 Prerequisites. We assume some familiarity with complex analysis, basic notions of topology, (and not very advanced) differential geometry.

We recap some important ideas from complex analysis.

Definition 0.1. A function $f : U \to \mathbb{C}$ defined on an open $U \subseteq \mathbb{C}$ is *holomorphic* if for all $z_0 \in U$, the limit

$$f'(z_0) \coloneqq \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Equivalently, writing $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ for real-valued functions u, v, the function f is holomorphic if it is smooth and satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}.$$

Theorem 0.2. *A function $f : U \to \mathbb{C}$ is holomorphic if and only if it is analytic, i.e. for all $z_0 \in U$ there is a neighbourhood $U' \subseteq U$ of z_0 such that $f|_{U'}$ is given by a convergent power series $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$.*

In particular, polynomials in z are holomorphic. The proof of the theorem above uses the following important result.

Theorem 0.3 (Cauchy's integral formula). *Suppose $f : U \to \mathbb{C}$ is holomorphic and write $D = \{z \mid |z - z_0| \leqslant r\} \subset U$. Then for all a in the interior of D, we have*

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - a} dz$$

where the integral is over the boundary of D (a circle oriented anticlockwise).

Corollary 0.4 (Identity principle). *Given two holomorphic functions $f, g : U \to \mathbb{C}$ on an open connected domain $U \subseteq C$, if we have $f|_V = g|_V$ for a non-empty open set $V \subseteq U$ then we have $f = g$.*

§1 Riemann surfaces

The definition of a Riemann surface will be modelled after that of a smooth manifold, except that we will use holomorphic transition functions instead.

Definition 1.1. A Riemann surface is a Hausdorff topological space X with an open cover $X = \bigcup_\alpha U_\alpha$, and for each α a homeomorphism $\phi_\alpha : U_\alpha \to \widetilde{U}_\alpha$ to an open subset $\widetilde{U}_\alpha \subset \mathbb{C}$ such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the function $\phi_\alpha \circ \phi_\beta^{-1}$ is holomorphic where defined. We say each pair (U_α, ϕ_α) is a *chart*, and the collection of all charts forms an *atlas*.

Remark 1.2. A higher dimensional complex manifold is analogously defined, where we ask for the charts to go into $\mathbb{C}^n$ with transition maps holomorphic in each coordinate.

In practice, we work locally. For $p \in X$, p is in some open U_α and then ϕ_α is a complex-valued function on U_α which we write as z . Thus locally we are doing complex analysis. The transition functions being holomorphic ensures this local coordinate is well-defined up to a holomorphic function, i.e. if another chart provides a local coordinate w then $w = f(z)$ for some holomorphic invertible function f.

Definition 1.3. A function $f : X \to \mathbb{C}$ is called ***holomorphic*** if $f|_{U_\alpha}$ is holomorphic (in the local coordinate) for all charts $U_\alpha \subset X$.

As opposed to differential geometry where a manifold has many smooth functions on it, a Riemann surface might have no non-constant holomorphic functions (eg. if it is compact). On the other hand, maps between Riemann surfaces are usually abundant.

Definition 1.4. Let X be a Riemann surface with atlas $\{(U_\alpha, \phi_\alpha)\}$ and Y a Riemann surface with atlas $\{(V_\alpha, \phi_\alpha)\}$. A map $f : X \to Y$ is ***holomorphic*** if for all pairs of indices α, β, the map $\phi_\beta \circ f \circ \phi_\alpha^{-1}$ is holomorphic where defined.

We say a bijection f is a biholomorphism if f^{-1} is holomorphic, and we usually treat biholomorphic Riemann surfaces as identical.

Example 1.5. The following constructions are standard.

1. The complex plane $\mathbb{C}$ itself is a complex manifold, with a single chart (with the identity map). Likewise any open $U \subset \mathbb{C}$ is a complex manifold with a single chart.
2. The 2-sphere $S^2 = \mathbb{C} \cup \{\infty\}$ is a complex manifold called the ***Riemann sphere***, with two charts $S^2 \setminus \{\infty\} \cong \mathbb{C}$ (via the identity) and $S^2 \setminus \{0\} \cong \mathbb{C}$ (via $z \mapsto \frac{1}{z}$).
3. If $\pi : S \to X$ is a topological covering space and X is a Riemann surface, then S canonically obtains the structure of a Riemann surface that makes π holomorphic.
4. Conversely we can construct Riemann surfaces as quotients– consider a lattice $\Lambda \subset \mathbb{C}$ (i.e. a rank two additive subgroup generated by two $\mathbb{R}$-linearly independent complex numbers). We can take the quotient topological space $\mathbb{C}/\Lambda$ which is homeomorphic to a torus, and equip it with charts on which each open set is small enough to lie within a fundamental domain of the Λ-action after pulling back to $\mathbb{C}$. This is an ***elliptic curve***.

We will now discuss two important classes of examples in detail.

§1.1 Affine algebraic curves

§1.1 Affine algebraic curves. Take a polynomial $p(z, w) : \mathbb{C}^2 \to \mathbb{C}$ in two complex variables, and consider the set $S = \{(z, w) \in \mathbb{C}^2 \mid p(z, w) = 0\}$. This is an ***algebraic curve***. When is this a Riemann surface? This has to do with whether the space is singular or not– in particular we want to be able to locally write w as a function of z (or vice versa). The implicit function theorem deals with precisely this.

Theorem 1.6 (Implicit function theorem). ***For*** p, S ***as above and*** $(z_0, w_0) \in S$, ***if*** $\frac{\partial p}{\partial w}(z_0, w_0) \neq 0$ ***then there are open balls*** $B_1 \ni z_0$ ***and*** $B_2 \ni w_0$ ***in*** $\mathbb{C}$ ***such that*** $S \cap (B_1 \times B_2) \subseteq \mathbb{C}^2$ ***is the graph of a holomorphic function*** $\sigma : B_1 \to B_2$, ***i.e.*** $S \cap (B_1 \times B_2) = \{(z, \sigma z) \mid z \in B_1\}$. ***In particular,*** (z_0, w_0) ***has a neighbourhood in*** S ***that is homoeomorphic to*** B_1.

Proof. We may assume $(z_0, w_0) = (0, 0)$. First consider the holomorphic function $p(0, w)$, this is a polynomial in one variable with $p(0, w) = 0$ for $w = 0$ and $\frac{\partial p}{\partial w}(0, 0) \neq 0$ by hypothesis. Thus 0 is a simple zero. As zeros of holomorphic functions are isolated, we can find a $\delta_2 > 0$ with $p(0, w) \neq 0$ whenever $0 < |w| < 2\delta_2$. By continuity of p, there is a $\delta_1 > 0$ such that $p(z, w) \neq 0$ whenever $|w| = \delta_2$, $|z| \leqslant \delta_1$. Let B_1, B_2 be the open balls around the origin of radii δ_1, δ_2 respectively. Then for $(z, w) \in S \cap (B_1 \times B_2)$, will write w as a function of z.

For this, we use that for holomorphic f and a simple closed curve γ, we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(w)}{f(w)} dw = \#\{\text{zeros of } f \text{ in the region bounded by } \gamma.\}$$

Consider the holomorphic functions $f_z(w) = p(w, z)$ for $z \in B_1$, and the contour $\gamma(t) = \delta_2 \cdot e^{2\pi i t}$. Then $N(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_z'(w)}{f_z'(w)} dw$ is an integer that varies continuously with z, and so considering $z = 0$ gives us

$N(z) = 1$ identically. Thus for each $z \in B_1$ there is a unique $w \in B_2$ with $f_z(w) = 0$. In fact the zero is then given by

$$\sigma(z) = \frac{1}{2\pi i} \int_\gamma w \frac{f_z'(w)}{f_z(w)} dw,$$

which expresses w as a holomorphic function of z as required. □

Corollary 1.7. *For p, S as above, $S \subset \mathbb{C}^2$ is a Riemann surface if for all $(z_0, w_0) \in S$, either $\frac{\partial p}{\partial z}(z_0, w_0) \neq 0$ or $\frac{\partial p}{\partial w}(z_0, w_0) \neq 0$. In this case, we say S is the* affine algebraic curve *defined by* p.

§1.2 Projective curves. Recall the projective plane is $\mathbb{P}^2 = \mathbb{C}^3 \setminus \{0\}/\sim$ where $(z_1, z_2, z_3) \sim (\lambda z_2, \lambda z_2, \lambda z_3)$ for all $\lambda \in \mathbb{C}^\times$. This is compact– this fact is easily seen by observing that $\mathbb{P}^2$ is the quotient of the (compact) subset of unit vectors in $\mathbb{C}^3$ by the action of the compact group $S^1 \subset \mathbb{C}^\times$. Points in $\mathbb{P}^2$ can be labelled by the homogeneous coordinates $[z_0 : z_1 : z_2]$, likewise for higher dimensional projective spaces.

Write $U_0 = \{[z_0 : z_1 : z_2] \mid z_0 \neq 0\}$ and note that this is homeomorphic to $\mathbb{C}^2$ since all points have homogeneous coordinates of the form $[1 : z : w]$. The complement $\mathbb{P}^2 \setminus U_0$ is homeomorphic to $\mathbb{P}^1$, so we can say $\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{C} \cup \{\infty\}$.

Note polynomials in z_0, z_1, z_2 don't take well-defined values on $\mathbb{P}^2$, but the vanishing locus of a homogeneous polynomial is nonetheless well-defined. Indeed, if $p(z_0, z_1, z_2)$ is homogeneous of degree d then for $\lambda \in \mathbb{C}^\times$ we have $p(\lambda z_0, \lambda z_1, \lambda z_2) = \lambda^d p(z_0, z_1, z_2)$. Thus it makes sense to ask when a homogeneous polynomial p defines a Riemann surface inside $\mathbb{P}^2$.

Proposition 1.8. *If $p(z_0, z_1, z_2)$ is a homogeneous polynomial such that the only solution to $\frac{\partial p}{\partial z_0} = \frac{\partial p}{\partial z_1} = \frac{\partial p}{\partial z_2} = 0$ is $(0, 0, 0)$, then the vanishing set S of p inside $\mathbb{P}^2$ is naturally a compact Riemann surface.*

Proof. We will show that $S \cap U_0 \subset \mathbb{C}^2$ is a Riemann surface by using the implicit function theorem. It will be true similarly and in a compatible way for U_1 and U_2, giving the required charts. The compactness then follows since $\mathbb{P}^2$ is compact and $S \subset \mathbb{P}^2$ is closed.

To use the implicit function theorem, we need to rule out the possibility that there is a $[1 : z : w] \in S$ where $\frac{\partial p}{\partial z_1}(1, z, w) = \frac{\partial p}{\partial z_2}(1, z, w) = 0$. Note since p is homogeneous, say of degree d, we have Euler's identity $\sum_{i=0}^2 z_i \frac{\partial p}{\partial z_i} = d \cdot p$. Thus if $\frac{\partial p}{\partial z_1}(1, z, w) = \frac{\partial p}{\partial z_2}(1, z, w) = 0$ we must have $\frac{\partial p}{\partial z_0}(1, z, w) = 0$, contradicting the hypothesis that $(0, 0, 0)$ is the only point where this happens. □

Example 1.9. Let $p(z, w) = z^5 + w^5 - zw - 1$. We homogenize to get $p(z_0, z_1, z_2) = z_1^5 + z_2^5 - z_0^3 z_1 z_2 - z_0^3$, with vanishing set $S \subset \mathbb{P}^2$. Then $S \cap U_0$ is the vanishing locus of the original polynomial, while $S \cap (\mathbb{P}^2 \setminus U_0)$ has five points $[0 : 1 : e^{\frac{-2\pi i k}{5}}]$, $k = 0, \ldots, 4$.

Remark 1.10. One can show that a Riemann surface is always orientable, i.e. there is an atlas where all the Jacobians of the transition functions have positive determinant. Thus compact Riemann surfaces are given by adding a holomorphic structure to an n-holed torus, $n \geqslant 0$. Donaldson sketches a proof of this using morse theory.

Remark 1.11. Historically Riemann surfaces arose as the natural domains of 'germs' of multi-valued holomorphic functions. We won't go into this.

§2 Holomorphic maps

Holomorphic functions are well understood, so we can be ambitious and aim to have a structure theory of maps between Riemann surfaces (which are locally holomorphic). This can shed light on important structure of the Riemann surfaces– for instance a compact Riemann surface has no non-constant holomorphic maps to $\mathbb{C}$. By contrast, affine curves $S \subset \mathbb{C}^2$ have many holomorphic functions.

Lecture 2, 17/10/23

Example 2.1 (Maps between torii). Consider $\mathbb{C}/\Lambda$ for Λ a lattice. When does a linear map $G : \mathbb{C} \to \mathbb{C}$ induce a holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$? Clearly we need $[G(z)] = [G(z + \lambda)] \in \mathbb{C}/\Lambda$ for all $\lambda \in \Lambda$, so if $G(z) = az + b$ then this amounts to the requirement that $a\Lambda \subset \Lambda$. Thus for instance $a = 1$ works, i.e. translations always induce maps $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$. For a non-trivial example, consider $\Lambda = \langle 1, i \rangle$ and $G(z) = (1 + i)z$. This gives a $2 : 1$ cover from the torus to itself.

Example 2.2. Just like holomorphic functions, we have a notion of *meromorphic functions* on a Riemann surface i.e. $f : X \to \mathbb{C} \cup \{\infty\}$ that are not identically ∞ such that for all charts $\psi : U \to \psi(U) \subseteq \mathbb{C}$, the function $f \circ \psi^{-1}$

is meromorphic on $\psi(U)$. Viewing $\mathbb{C} \cup \infty$ as the Riemann sphere, we see that meromorphic functions are just holomorphic maps to $\mathbb{P}^1$.

To get general theory, we use the inverse function theorem.

Theorem 2.3 (Inverse function theorem). *Let $U \ni 0$ be an open subset of $\mathbb{C}$, and $f : U \to \mathbb{C}$ holomorphic such that $f'(0) \neq 0$. Then there is a possibly smaller neighbourhood $U' \subseteq U$ of 0 such that $f : U' \to f(U')$ has a holomorphic inverse.*

Proof. Similar to implicit function theorem. □

Corollary 2.4. *Suppose $U \ni 0$ is an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a holomorphic function that vanishes on 0 but is not identically zero. Then there is some $k \geqslant 1$ such that on a possibly smaller neighbourhood $U' \subseteq U$ of 0, we have $f(z) = g(z)^k$ for a holomorphic function $g : U' \to \mathbb{C}$ with $g'(0) \neq 0$.*

Proof. This is automatic– since f is analytic and vanishes at the origin, we have $f(z) = z^k(a_0 + a_1 z + \dots)$ for $k \geqslant 1$, $a_0 \neq 0$. Thus $f(z) = z^k h(z)$ for some holomorphic function h with $h(0) \neq 0$. Then by the inverse function theorem applied to $z \mapsto z^k$, the function $z \mapsto z^{1/k}$ is well-defined on a neighbourhood of $h(0)$ and we can set $g(z) = z \cdot h(z)^{1/k}$ to get the required result. □

Proposition 2.5. *Suppose $F : X \to Y$ is a non-constant holomorphic map between connected Riemann surfaces. Then for all $x \in X$, there exist neighbourhoods $U \ni x$, $V \ni f(x)$ and biholomorphisms $\phi : U \to \phi(U) \subseteq \mathbb{C}$, $\psi : U \to \psi(U) \subseteq \mathbb{C}$ such that $\phi(x) = 0$ and $\psi \circ F \circ \phi^{-1}$ is given by $z \mapsto z^k$ for some integer $k \geqslant 1$.*

Proof. This is a consequence of the above results. Start with any charts ϕ', ψ' around $x, f(x)$ respectively and set $f = \psi' \circ F \circ \phi'^{-1}$. This is a holomorphic map, and after composing the charts with translations we may assume $f(0) = 0$. Then $f(z) = g(z)^k$ for some holomorphic g satisfying $g(0) = 0$, $g'(0) \neq 0$. Define $\phi = g \circ \phi'$, and note this is holomorphic and invertible on a possibly smaller open neighbourhood of x. Thus we have a new chart in which $\psi' \circ F \circ \phi^{-1}$ has the required form. □

Note the k in the above proposition depends on both F and x, but is independent of the choice of charts. To emphasise the dependence on x we will often write k_x.

Definition 2.6. In the situation of the theorem above, we say $x \in X$ is a *ramification point* of F if $k_x > 1$. The subset of all ramification points $R \subseteq X$ is called the *ramification locus*, and its image $F(R) \subseteq Y$ is called the *branch locus*.

Since the ramification locus R is defined locally by $x \in R \iff F'(x) = 0$ and since F' is holomorphic in each chart, we see that R is discrete (and hence finite if X is compact). Likewise if Y is compact (or F is proper), then the branch locus $F(R)$ is finite (resp. discrete).

Now assume F is proper, and consider the preimage $F^{-1}(y) \subset X$ of a point $y \in Y$. This set is discrete and compact, hence is finite. So we can count it. Define (temporarily) the *degree of F at y* to be

$$d(y) = \sum\nolimits_{x \in F^{-1}(y)} k_x.$$

This is a temporary definition because in fact it does not depend on y at all. Thus we define the *degree of F* to be its degree at any $y \in Y$.

Proposition 2.7. *The integer $d(y)$ does not depend on y.*

Proof. We will show the function $d : Y \to \mathbb{Z}$ is locally constant on Y, hence constant since Y is connected. Given $y \in Y$ with preimages $x_1, \dots, x_n \in X$, pick neighbourhoods $V \ni Y$, $U_i \ni x_i$ sufficiently small so that $U_i \cap U_j = \emptyset$ for $i \neq j$, $F(U_i) \subseteq V$, and the restriction $F : U_i \to V$ is given by $z \mapsto z^{k_i}$ in local coordinates. Here $k_i = k_{x_i}$, and $i = 1, \dots, n$. We wish to find a possibly smaller neighbourhood $W \ni y$ on which d is constant.

Claim that there is a neighbourhood $W \ni y$ such that $F^{-1}(W) \subseteq U_1 \cup U_2 \cup \dots \cup U_n$. The properness hypothesis is essential here. Indeed, the natural map $\mathbb{C} \sqcup \mathbb{P}^1 \to \mathbb{P}^1$ is not proper and so while $\infty \in \mathbb{P}^1$ has one pre-image, any point in the neighbourhood of ∞ has two. Now having seen why the claim is non-trivial, we will prove it using point-set topology arguments. Choose a sequence of open sets $V_1 \supseteq V_2 \supseteq \dots \ni y$ such that $\bigcap \overline{V_i} = \{y\}$. If the claim were false, then in particular $W = V_i$ does not suffice and we can choose a point $z_i \in F^{-1}(V_i) \setminus (U_1 \cup \dots \cup U_n)$ for each V_i. This gives a sequence $z_1, z_2, \dots$ in the set $F^{-1}(\overline{V_1}) \setminus (U_1 \cup \dots \cup U_n)$, which by the properness hypothesis is compact. Thus there is a subsequence converging to some $z \notin U_1 \cup \dots \cup U_n$.

But by continuity of F and choice of V_i , we have $F(z) = y$. This is a contradiction since we know all pre-images of y are contained in some U_i.

Having proved the claim, the result follows since for any $y' \in W$, we have

$$d(y') = \sum_{x \in F^{-1}y'} k_x = \sum_{i=1}^{n} \sum_{x \in F^{-1}y' \cap U_i} k_x$$

but locally on U_i, the map is $z \mapsto z^{k_i}$ and hence we have $\sum_{x \in F^{-1}y' \cap U_i} k_x = k_i$ since either $y' = y$ (and it has one preimage in U_i) or $y' \neq y$ (and it has k_i simple preimages). □

Corollary 2.8. *Any proper non-constant holomorphic map between Riemann surfaces is surjective.*

Remark 2.9. The above corollary could also have been obtained from the open mapping principle.

We next use this to obtain topological consequences– in particular, counting poles of meromorphic functions leads to a classification of compact Riemann surfaces.

Corollary 2.10. *Let X be a compact Riemann surface. If there is a meromorphic function $f : X \to \mathbb{C} \cup \{\infty\}$ with a single simple pole, then f provides a biholomorphism $X \cong \mathbb{P}^1$.*

Proof. Note we can view f as a holomorphic map $X \to \mathbb{P}^1$. Since X is compact, f is proper and hence has degree $d = d(\infty) = 1$. Thus F is a continuous bijection between compact Hausdorff spaces, so by the topological inverse function theorem, f is a homeomorphism. To show its a biholomorphism it suffices to show the inverse continuous map f^{-1} is in fact holomorphic. But this is clear since in a local chart around any $x \in X$, f has non-vanishing derivative and so the local inverse is holomorphic by the inverse function theorem. □

This concludes for now the study of maps between Riemann surfaces. In higher dimensions, there are many more interesting maps but codimension one phenomena are nonetheless controlled by the same rules which describe maps between Riemann surfaces.

§ 3 Calculus on surfaces

We wish to eventually develop a theory of holomorphic differential forms and de Rham cohomology on Riemann surfaces. Before that, we give an account of the differential geometry of smooth surfaces.

§ 3.1 Tangent and cotangent spaces. There are various ways to define vector fields and differential forms on surfaces, for instance these can be realised most naturally as sections of the tangent and cotangent bundles respectively. Here we will take a more abstract approach, defining tangent vectors as infinitesimal paths and differential forms as derivations of smooth functions.

Lemma 3.1. *Let $U \subseteq \mathbb{R}^2$ be an open neighbourhood of 0, $f : U \to \mathbb{R}$ a smooth function, and $\gamma_1, \gamma_2 : \mathbb{R} \to U$ two smooth paths with $\gamma_1(0) = \gamma_2(0) = 0$. Let $\chi : U \to V \subseteq \mathbb{R}^2$ be a diffeomorphism with $\chi(0) = 0$, and set $\tilde{f} = f \circ \chi^{-1}$, $\tilde{\gamma}_1 = \chi \circ \gamma_1$, $\tilde{\gamma}_2 = \chi \circ \gamma_2$. Then the following hold.*

(i) If both partial derivatives $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ vanish at 0 then the same is true for $\tilde{f}$.

(ii) If the derivatives $\frac{d\gamma_1}{dt}(0)$ and $\frac{d\gamma_2}{dt}(0)$ agree then the same is true for $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$.

Proof. These are straightforward consequences of the chain rule. □

Given the diffeomorphism invariance of these notions, we can make the following definitions. Let S be a (real, smooth) surface with a point $p \in S$, f a smooth real-valued function defined on a neighbourhood of p, and $\gamma_1, \gamma_2 : \mathbb{R} \to S$ smooth paths with $\gamma_1(0) = \gamma_2(0) = p$.

Definition 3.2. We say f is *constant to first order* at p if $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ both vanish at p in some local chart. Likewise we say γ_1, γ_2 are *equal to first order* at p if their derivatives agree in a local chart.

Definition 3.3. The *cotangent space* T_p^*S of S at p is the set of equivalence classes of smooth functions on open neighbourhoods of $p \in S$, where f_1 and f_2 are equivalent if and only if $f_1 - f_2$ is constant to first order at p.

Dually, the *tangent space* T_pS of S at p is the set of equivalence classes of smooth paths $\gamma : \mathbb{R} \to S$ with $\gamma(0) = p$ where two paths are equivalent if they agree to first order at p.

Note there is an obvious map $C^\infty(S) \to T_p^*S$ sending a function f to its equivalence class, for which we will write $[df]_p$ (and omit the brackets whenever convenient). This gives T_p^*S a natural vector space structure, and in fact we can check that it is two dimensional. Indeed choosing local coordinates x_1, x_2 around p, these provide smooth functions and so we have two classes $[dx_1]_p, [dx_2]_p \in T_p^*S$. One checks that

$$[df]_p = \frac{\partial f}{\partial x_1}(p) \cdot [dx_1]_p + \frac{\partial f}{\partial x_2}(p) \cdot [dx_2]_p,$$

hence any choice of local coordinates provides a basis for the cotangent space at the point. Similarly T_pS is a vector space, by choosing linear representatives of paths.

The two vector spaces are naturally dual to each other– if $\gamma : \mathbb{R} \to S$ is a path with $\gamma(0) = p$ and f is a smooth function defined on a neighbourhood of p, then the derivative $(f \circ \gamma)'(0) \in \mathbb{R}$ depends only on the classes $[df]_p \in T_p^*S$, $[\gamma] \in T_pS$. Thus we have a perfect bilinear pairing

$$T_pS \times T_p^*S \to \mathbb{R}, \qquad ([\gamma], [f]) \mapsto (f \circ \gamma)'(0)$$

giving an isomorphism $T_p^*S = \mathrm{Hom}_{\mathbb{R}}(T_pS, \mathbb{R})$.

As a set, the *cotangent bundle* of S is given by $T^*S = \bigsqcup_p T_p^*S$. This can naturally be given the structure of a vector bundle over S, but we won't go into the details of this. Smooth sections of this bundle are called 1-forms, but since we did not give the vector bundle structure on T^*S we will define what it means to be a 1-form explicitly.

Definition 3.4. A smooth 1-form α on S is a map $\alpha : S \to T^*S$ satisfying $\alpha(p) \in T_p^*S$ for all p, and varying smoothly with p in the following sense: in local coordinates (x_1, x_2) around any $p \in S$, we can write $\alpha = \alpha_1 dx_1 + \alpha_2 dx_2$ for smooth functions α_1, α_2.

This notion of smoothness is well-defined. Indeed if (y_1, y_2) were another pair of local coordinates around $p \in S$, then the corresponding cotangent vectors are related as

$$dx_i = \frac{\partial x_i}{\partial y_1} dy_1 + \frac{\partial x_i}{\partial y_2} dy_2$$

wherever defined so the coefficients of the 1-form α remain smooth.

(Co)tangent vectors behave functorially with respect to smooth maps between surfaces– a map $F : S \to Q$ naturally induces linear maps

$$\begin{aligned} dF_p &: T_pS \to T_{F(p)}Q, \quad [\gamma] \mapsto [F \circ \gamma] \\ F_p^* &: T_{F(p)}^*Q \to T_p^*S, \quad [f] \mapsto [f \circ F]. \end{aligned}$$

Note the behaviour is covariant on tangent vectors and contravariant on cotangent vectors. This pullback map extends to 1-forms naturally, so if α is a 1-form on Q then the 1-form $F_p^*\alpha$ given by $(F_p^*\alpha)(p) = F_p^*(\alpha(F(p)))$ is still smooth.

Lecture 3, 24/10/23

§ 3.2 Integration along curves. One of the things that makes differential forms useful is that they can be integrated along curves. In the simplest case, suppose $\gamma : [0, 1] \to S$ is a smooth curve in our surface whose image lies in a chart with local coordinates x_1, x_2 and suppose α is a 1-form given on this chart as $\alpha_1 dx_1 + \alpha_2 dx_2$. Writing $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ in these local coordinates, we can define the integral of α along γ as

$$\int_\gamma \alpha = \int_0^1 (\alpha_1 \cdot \gamma_1'(t) + \alpha_2 \cdot \gamma_2'(t)) dt.$$

One checks that this is independent of local coordinates chosen. For a general curve γ, we cut it into smaller curves $\gamma_1, ..., \gamma_n$ each of which is contained in a single coordinate chart, and then define $\int_\gamma \alpha = \sum_{i=1}^n \int_{\gamma_i} \alpha$.

The fundamental theorem of calculus holds i.e. $\int_\gamma df = f(1) - f(0)$ for any smooth function f. This can be proven locally on charts.

§ 3.3 Higher differential forms. If we wish to analogously extend the theory of integration to surfaces, we need to define 2-forms. Just as a function gives a 1-form, there is an operator taking a 1-form to a 2-form. But this is more involved; the classical motivation for the definition is as follows.

Given a 1-form α on a surface S, we ask when is $\alpha = df$ for some function f? When $S = \mathbb{R}^2$, we have $\alpha = \alpha_1 dx_1 + \alpha_2 dx_2$ for a choice of coordinates. If $\alpha = df$ then $\frac{\partial f}{\partial x_1} = \alpha_1$, $\frac{\partial f}{\partial x_2} = \alpha_2$. Thus by the symmetry of mixed partial derivatives, a necessary condition for α to be of the form df is that the function $R = \frac{\partial \alpha_1}{\partial x_2} - \frac{\partial \alpha_2}{\partial x_1}$ vanishes identically. This is also sufficient– if $R = 0$ then we can find f as follows. Define functions f_1, f_2 by

$$f_2(x_1, x_2) = \int_0^{x_1} \alpha_1(t, 0) + \int_0^{x_2} \alpha_2(x_1, t)dt, \qquad f_1(x_1, x_2) = \int_0^{x_2} \alpha_1(0, t) + \int_0^{x_1} \alpha_2(t, x_2)dt.$$

By construction, $\frac{\partial f_i}{\partial x_i} = \alpha_i$. Then Green's theorem applied to the rectangle V with vertices $(0,0)$, $(x_1, 0)$, $(0, x_2)$, (x_1, x_2) shows $f_1(x_1, x_2) - f_2(x_1, x_2) = \int_V R dx_1 dx_2$. Thus if R vanishes, then $f_1 = f_2$ and this is the required function f.

We generalise this to arbitrary surfaces, with the motivation that the problem in question is controlled by some anti-symmetric function. With this in mind, recall that for a real vector space E we define $\bigwedge^2 E^*$ to be the space of skew-symmetric bilinear maps $E \times E \to \mathbb{R}$. There is a natural wedge-product operator

$$\begin{aligned} \wedge : E^* \times E^* &\to \textstyle\bigwedge^2 E^* \\ (\alpha, \beta)(e, f) &= \alpha(e)\beta(f) - \alpha(f)\beta(e). \end{aligned}$$

It is clear that this product is linear in each variable and $\alpha \wedge \beta = -\beta \wedge \alpha$. If E is 2-dimensional, one checks that $\bigwedge^2 E^*$ is 1-dimensional and any basis α_1, α_2 of E^* induces a basis $\alpha_1 \wedge \alpha_2$ of $\bigwedge^2 E^*$.

We are interested in the case $E = T_p S$ for a smooth surface S, so E^* is the cotangent space $T_p^* S$. A choice of local coordinates x_1, x_2 induces a basis element $dx_1 \wedge dx_2$ for $\bigwedge^2 T_p^* S$.

Definition 3.5. A (smooth) 2-form ρ on S is a map from S to $\bigcup_{p \in S} \bigwedge^2 T_p^* S$ such that $\rho(p) \in \bigwedge^2 T_p^* S$ for all $p \in S$, and such that ρ varies smoothly in the sense that locally we have $\rho = R(x_1, x_2) dx_1 \wedge dx_2$ for some smooth function R.

This is well defined independent of chart; for a different choice of coordinates y_1, y_2, ρ has the form $\rho = R(y_1, y_2) \cdot J(y_1, y$ where $J(y_1, y_2)$ denotes the Jacobian of y_1, y_2 with respect to x_1, x_2, and $dy_i = \frac{\partial y_i}{\partial x_1} dx_1 + \frac{\partial y_i}{\partial x_2} dx_2$.

One can pull back 2-forms just like one does for 1-forms. In particular if $F : S \to Q$ is a smooth map and ρ is a 2-form on Q then one obtains a 2-form $F^*\rho$ on S, expressed in local coordinates in a very similar way.

§3.4 The exterior derivative. We write $\Omega^i(S)$ for the set of smooth i-forms on S, where a smooth 0-form is nothing but a function. We have already seen that there is a differential $d : \Omega^0(S) \to \Omega^1(S)$ sending a function f to df. The content of the following result is that this extends nicely.

Lemma 3.6. *There is a unique way to define an $\mathbb{R}$-linear map $d : \Omega^1(S) \to \Omega^2(S)$ such that*

(i) For $\alpha \in \Omega^1(S)$ and open set $U \subset S$, we have $\alpha|_U \in \Omega^1(U)$ and $(d\alpha)|_U = d(\alpha|_U)$.

(ii) If $\alpha_1 = \alpha_2$ on an open set $U \subset S$, then $d\alpha_1 = d\alpha_2$ on U.

(iii) For any function $f \in \Omega^0(S)$, we have $d(df) = 0$.

(iv) If $f \in \Omega^0(S)$ and $\alpha \in \Omega^1(S)$, we have $d(f \cdot \alpha) = df \wedge \alpha + f \cdot d\alpha$.

Proof. If we have have defined d with these properties, then for a 1-form $\alpha = \alpha_1 \cdot dx_1 + \alpha_2 \cdot dx_2$ in local coordinates on an open $U \subset S$, we have

$$\begin{aligned} d\alpha &= d(\alpha_1 \cdot dx_1 + \alpha_2 \cdot dx_2) \\ &= d\alpha_1 \wedge dx_1 + d\alpha_2 \wedge dx_2 \\ &= \ldots \\ &= \left(\frac{\partial \alpha_2}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_2} \right) dx_1 \wedge dx_2. \end{aligned}$$

This shows uniqueness, and also gives a way to define d after one checks it is independent of charts. $\square$

Having expressed 2-forms as derivatives, we now see how to integrate them. For this we need to assume S is oriented (so there are charts such that the coordinate-change Jacobians are everywhere positive). Let $\rho \in \Omega^2(S)$ have compact support in a single chart. Then we can write $\rho = R(x_1, x_2) dx_1 \wedge dx_2$ in local coordinates, and define

$$\int_S \rho = \int_{\mathbb{R}^2} R(x_1, x_2) \cdot dx_1 dx_2$$

where the right hand side is a Lebesgue integral. If y_1, y_2 is another oriented chart with positive coordinate-change Jacobian, then that this is well-defined is the usual transformation law for multiple integrals.

To integrate in general, we need the following technical lemma which says partitions of unity exist.

Lemma 3.7. *Let* $K \subset S$ *be compact and let* $U_1, \ldots, U_n$ *be open sets with* $K \subseteq U_1 \cup \ldots \cup U_n$. *Then there are smooth non-negative functions* $f_1, \ldots, f_n$ *on* S *with* $f_1 + \ldots + f_n = 1$ *on* K, *such that each* f_i *is supported on* U_i.

Definition 3.8. For a 2-form $\rho \in \Omega^2(S)$ with compact support, we define $\int_S \rho = \sum_i \int_S (f_i \cdot \rho)$ where the f_i form a partition of unity over $\mathrm{Supp}(\rho)$ and each f_i is supported on a coordinate chart.

To see this makes sense, since $\sum_i f_i = 1$ we have $\rho = \sum_i (f_i \cdot \rho)$ and each $f_i \cdot \rho$ is supported on a coordinate chart so can be integrated locally. The integral is independent of the choice of partition of unity by linearity of Lebesgue integrals.

Theorem 3.9 (Stokes' theorem). *If* α *is a compactly supported* 1*-form on an oriented surface* S *with boundary* ∂S, *then* $\int_{\partial S} \alpha = \int_S d\alpha$.

Note we haven't defined manifolds with boundary but a precise definition with nice pictures can be found on Wikipedia. Locally, the boundary plays the same role as the curve in Green's theorem. Since the result is local, we can reduce to this case.

Thus on an oriented surface S, we have spaces of $0, 1, 2$-forms $\Omega^k(S)$ and exterior derivative operators $d : \Omega^k(S) \to \Omega^{k+1}(S)$. We also know how to integrate k-forms on k-dimensional submanifolds ($k \geqslant 1$), and Stoke's theorem which relates the two integrals. Lastly, we have the wedge product $\wedge : \Omega^1(S) \times \Omega^1(S) \to \Omega^2(S)$.

§ 3.5 The de Rham complex. Let S be a smooth surface, and consider the sequence of maps

$$0 \to \Omega^0(S) \xrightarrow{d} \Omega^1(S) \xrightarrow{d} \Omega^2(S) \to 0.$$

We know $d \circ d = 0$ so this is a chain complex, and we can define the *de Rham cohomology groups* as the cohomology spaces of this complex explicitly given by

$$\begin{aligned} H^0(S) &= \ker(d : \Omega^0(S) \to \Omega^1(S)) \\ H^1(S) &= \frac{\ker(d : \Omega^1(S) \to \Omega^2(S))}{\mathrm{im}(d : \Omega^0(S) \to \Omega^1(S))} \\ H^2(S) &= \frac{\Omega^2(S)}{\mathrm{im}(d : \Omega^1(S) \to \Omega^2(S))}. \end{aligned}$$

We say a k-form α is *closed* if $d\alpha = 0$ and *exact* if $\alpha = d\beta$ for some $(k-1)$-form β. Then every exact form is closed and the de Rham cohomology measures 'failure of closed forms to be exact'.

In general, $H^0(S) = \mathbb{R}$ for a connected surface S. We will now compute some examples.

Lemma 3.10 (de Rham cohomology of $\mathbb{R}^2$). *We have* $H^0(\mathbb{R}^2) = \mathbb{R}$, *and all other cohomology groups vanish.*

Proof. Since $\mathbb{R}^2$ is connected, we know $H^0(\mathbb{R}^2) = \mathbb{R}$. The claim $H^1(\mathbb{R}^2) = 0$ follows from the criterion for a closed 1-form to be exact. Thus it remains to show $H^2(\mathbb{R}^2) = 0$. Take a closed 2-form $\rho = R(x_1, x_2)dx_1 \wedge dx_2$ and claim we can find functions α_1, α_2 with $R(x_1, x_2) = \frac{\partial \alpha_1}{\partial x_2} - \frac{\partial \alpha_2}{\partial x_1}$, which will show the required result. This can be done, for example, by setting $\alpha_2 = 0$ and $\alpha_1(x_1, x_2) = \int_0^{x_2} R(x_1, t)dt$. $\square$

Proposition 3.11. *Let* $S = \mathbb{S}^2$ *be the sphere. Then* $H^1(S) = 0$.

Proof. Suppose α is a closed 1-form on S. Take the standard charts $U, V \subseteq S$ given by removing the north and south poles from S respectively. Then $U, V \cong \mathbb{R}^2$ so we can find functions f_U, f_V on U, V respectively such that $df_U = \alpha|_U$, $df_V = \alpha|_V$. On $U \cap V$, we see that $f_U - f_V$ is locally constant hence is constant. Thus adding a constant function to f_V if necessary, we see that f_U and f_V agree on $U \cap V$ and hence define a global function $f \in \Omega^0(S)$ with $df = \alpha$. $\square$

Proposition 3.12. *Let* $T = \mathbb{S}^1 \times \mathbb{S}^1$ *be the torus. Then* $H^1(T)$ *is a two-dimensional vector space.*

Proof. Consider the standard angular coordinates $\vartheta, \varphi \in [0, 2\pi)$. Let γ_1, γ_2 be the standard circles corresponding to $\vartheta = 0$ and $\varphi = 0$ respectively, and consider the map

$$F : \Omega^1(T) \to \mathbb{R}^2, \qquad \alpha \mapsto \left(\int_{\gamma_1} \alpha, \int_{\gamma_2} \alpha \right).$$

This is linear and vanishes on exact 1-forms by the fundamental theorem of calculus, so we get a well-defined map $F : H^1(T) \to \mathbb{R}^2$. We will show F is injective and surjective.

If α is a closed 1-form such that $\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha = 0$, then writing $\alpha = P d\vartheta + Q d\varphi$ locally shows that for any fixed φ we have

$$\int_{\gamma_2} \alpha = \int_0^{2\pi} P(u, \varphi) du = 0$$

and hence the function $f(\vartheta, \varphi) = \int_0^{\vartheta} P(u, \varphi) du$ is well-defined and satisfies $\frac{\partial f}{\partial \vartheta} = P(\vartheta, \varphi)$. Thus $\tilde{\alpha} = \alpha - df$ is a closed 1-form of the form $Q \cdot d\varphi$. But $d\tilde{\alpha} = 0$ implies Q is constant and we see that $Q = 0$ since $\int_{\gamma_2} Q d\varphi = 0$. Thus $\alpha = df$ is exact.

To show surjectivity, we consider the 1-forms $d\vartheta$, $d\varphi$ which get mapped to $(1, 0)$ and $(0, 1)$ respectively. $\square$

Remark 3.13. In general, a similar argument shows $H^1(S) = \mathrm{Hom}(\Pi_1(S, p), \mathbb{R})$ where $\Pi_1(S, p)$ is the fundamental group of S for some base point p. In particular if Σ_g is the closed orientable surface of genus g then $H^1(\Sigma_g) \cong \mathbb{R}^{2g}$.

§ 3.6 Compactly supported cohomology. We next calculate $H^2(S)$. For this, we need to first modify the definition of de Rham cohomology when S is non-compact, by setting $\Omega^i_C(S) \subseteq \Omega^i(S)$ to be the subspace of compactly supported i-forms. The exterior derivative respects the compact support property, hence we can define $H^i_C(S)$ to be the i-th cohomology of the complex

$$0 \to \Omega^0_C(S) \xrightarrow{d} \Omega^1_C(S) \xrightarrow{d} \Omega^2_C(S) \to 0.$$

The main theorem is the following.

Theorem 3.14. *For any connected surface S, we have $H^2_C(S) = \mathbb{R}$.*

Proof. We prove this in two steps– first the special case $S = \mathbb{R}^2$, and then use this to handle general surfaces.

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In the case of $\mathbb{R}^2$, the proof is by explicit integration. We will consider the map $I : \Omega^2_C(\mathbb{R}^2) \to \mathbb{R}$ given by $I(\rho) = \int_{\mathbb{R}^2} \rho$. This is well-defined by the compact support property. Surjectivity follows since we can explicitly find a smooth function ψ with $I(\psi \cdot dx_1 \wedge dx_2) = \int_{\mathbb{R}^2} \psi \cdot dx_1 dx_2 = 1$. Thus it remains to show that the kernel of I is precisely the space of exact forms. One containment is obvious by Stokes' theorem, so suppose $I(\rho) = 0$ and write $\rho = R(x, y) dx \wedge dy$ where (up to translation) R is supported in some box $[0, N]^2$. We construct $\alpha \in \Omega^1_C(\mathbb{R}^2)$ with $d\alpha = \rho$ by integrating R cleverly.

First consider the 'strong special case' when $\int_0^N R(x, y) dx = 0$ for all y. Define $g(x, y) = \int_0^x R(t, y) dt$. Then $g(N, y) = 0$ so $g(x, y) dy$ is a compactly supported 1-form and its derivative is $R(x, y) dx \wedge dy$ as required. In general, consider $g(x, y) dy$. Then to reduce to the strong special case, we must manage the quantity $H(y) = \int_0^N R(x, y) dx$ which doesn't vanish in general. Take $b : \mathbb{R} \to \mathbb{R}$ a 'bump function' with compact support such that $\int_0^N b(x) dx = 1$. Since $H : \mathbb{R} \to \mathbb{R}$ also has compact support, so does $h(x, y) = b(x) H(y)$. Moreover,

$$\int_{\mathbb{R}^2} h(x, y) dx \wedge dy = \int_0^N H(y) dy = \int_{\mathbb{R}^2} R(x, y) dx \wedge dy.$$

Thus $\int_0^N (R(x, y) - h(x, y)) dx = 0$, i.e. the 2-form $(R(x, y) - h(x, y)) dx \wedge dy$ satisfies the hypotheses of the strong special case and is hence exact. So to complete the proof, we need to show $h(x, y) dx \wedge dy = d\beta$ for some compactly supported 1-form β. But reversing x, y we know that $\int_0^N h(x, y) dy = b(x) \int_0^N H(y) dy = 0$, so this again is the strong special case.

Thus we wrote ρ as a sum of two 2-forms, both of which satisfy the hypotheses of the strong special case and are hence exact.

For a general connected surface S, again consider the map $I : \Omega^2_C(S) \to \mathbb{R}$ given by $\rho \mapsto \int_S \rho$. Surjectivity and vanishing on exact forms follows similarly as before, so we must show that $\ker I$ is precisely the space of exact forms. Suppose $\rho \in \Omega^2_C(S)$ is such that $\int_S \rho = 0$. By the compact support property, there are finitely many contractible charts $U_1, ..., U_n$ such that ρ is supported on a connected open set $U_1 \cup U_2 \cup ... \cup U_n$. Since we know the result for $\mathbb{R}^2$, we know it holds for each U_i, so we show the result by induction on n.

Suppose $n > 1$. By connectedness of $\bigcup_i U_i$ we can arrange that $U_i \cap U_{i+1} \neq \emptyset$ for all i, so that $V_1 = U_1 \cup ... \cup U_{n-1}$ and $V_2 = U_n$ are connected opens. Then by induction hypothesis, any 2-form $\tilde{\rho}$ compactly supported on V_i satisfying $\int_S \tilde{\rho} = 0$ must be exact. So take a partition of unity f_1, f_2 such that $\mathrm{Supp}(f_i) \subset V_i$ and $f_1 + f_2 = 1$ on $\mathrm{Supp}(\rho)$. Then $\rho = \rho_1 + \rho_2$, where $\rho_i = \rho \cdot f_i$ supported on V_i . From $\int_S \rho = 0$, we obtain a constant $c = \int_S \rho_1 = -\int_S \rho_2$. Choose a 2-form σ supported on $V_1 \cap V_2$ such that $\int_S \rho = c$. Then setting $\tilde{\rho}_1 = \rho_1 - \sigma$ and $\tilde{\rho}_2 = \rho_2 + \sigma$, we see that by induction hypothesis $\tilde{\rho}_i = d\alpha_i$ for some α_1, α_2. Then for the 1-form $\alpha = \alpha_1 + \alpha_2$, we have $\rho = d\alpha$ as required. □

§ 4 Complex differential forms

In the previous section, S was a smooth surface with no additional structure. At a point $p \in S$ the cotangent space T_p^*S is defined so that a real-valued function $f : S \to \mathbb{R}$ induces a cotangent vector $[df]_p$ (seen as an $\mathbb{R}$-linear map $T_pS \to \mathbb{R}$). Likewise, we can consider the *complexified cotangent space* $T_p^*S^{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(T_PS, \mathbb{C})$ defined so that a complex-valued function $f : S \to \mathbb{C}$ induces a cotangent vector $[df]_P : T_pS \to \mathbb{C}$. This makes sense for any surface S.

We now specialise to Riemann surfaces, where there is a distinguished class of holomorphic functions. What does this extra structure buy us?

Definition 4.1. Let V be a real vector space. A *complex structure* on V is an $\mathbb{R}$-linear map $J : V \to V$ such that $J^2 = -\mathrm{Id}$. Given such a vector space with complex structure, we say an $\mathbb{R}$-linear map $A : V \to \mathbb{C}$ is *complex linear* if $A(Jv) = i \cdot A(v)$ for all $v \in V$ and it is *complex anti-linear* if $A(Jv) = -i \cdot A(v)$ for all $v \in V$.

The cotangent spaces of a Riemann surface are naturally equipped with a complex structure.

Lemma 4.2. *Given a Riemann surface S and a point $p \in S$, there is a unique way to define a complex structure on T_pS such that the differential of any holomorphic function is complex linear.*

Sketch proof. This is a consequence of the fact that on open sets in $\mathbb{C}$, the map sending a holomorphic function to its derivative is $\mathbb{C}$-linear. Choosing a local complex coordinate $z = x + iy$ around p, the complex structure $J : T_pS \to T_pS$ then is given by multiplication by $\sqrt{-1}$ i.e. a tangent vector determined by a path $\gamma : \mathbb{R} \to S$ gets sent to the tangent vector $[i \cdot \gamma]$. □

Let V be an $\mathbb{R}$-vector space with a complex structure. Considering the $\pm i$-eigenspaces, we have the following useful piece of linear algebra: any $\mathbb{R}$-linear map $A : V \to \mathbb{C}$ can be written as a sum of a complex linear map $A' = \frac{1}{2}(A - i \cdot A \circ J)$ and a complex anti-linear map $A'' = \frac{1}{2}(A + i \cdot A \circ J)$, thus giving an orthogonal decomposition of $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$.

In the setting of Riemann surfaces, this gives a decomposition $T_p^*S^{\mathbb{C}} = T_p^*S' \oplus T_p^*S''$ of the complexified cotangent space into a complex linear part (*the holomorphic cotangent space*) and an anti-linear part (*the anti-holomorphic cotangent space*) respectively. The complex structure on T_pS was chosen such that for any holomorphic function f we have $[df]_p \in T_p^*S'$ and $[d\bar{f}]_p \in T_p^*S''$.

§ 4.1 Decomposing 1-forms and exterior derivatives. The decomposition of the cotangent space extends to the space of smooth complex 1-forms naturally, and we write $\Omega^1(S, \mathbb{C}) = \Omega^{1,0}(S) \oplus \Omega^{0,1}(S)$. The exterior derivatives $d : \Omega^0(S, \mathbb{C}) \to \Omega^1(S, \mathbb{C})$ and $d : \Omega^1(S, \mathbb{C}) \to \Omega^2(S, \mathbb{C})$ decompose in a compatible fashion: there is a commutative square

$$\begin{tikzcd}
\Omega^{0,1}(S) \arrow[r, "-\partial(=-d)"] & \Omega^2(S, \mathbb{C}) \\
\Omega^0(S, \mathbb{C}) \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"] & \Omega^{1,0}(S) \arrow[u, "\bar{\partial}(=d)"']
\end{tikzcd}$$

where the *Dolbeault operators* $\partial, \bar{\partial}$ are given by composing d with the natural projections $\Omega^1(S, \mathbb{C}) \to \Omega^{0,1}(S), \Omega^{1,0}(S)$. In particular, we have $d = \partial + \bar{\partial}$ and $\partial^2 = \bar{\partial}^2 = 0$. The construction is explicit on a chart– choosing a local complex coordinate $z = x + iy$, the vectors $dz = dx + idy$ and $d\bar{z} = dx - idy$ give bases of T_p^*S' and T_p^*S''

respectively for p in the chart. Given a complex function f on this chart, we can write

$$df = \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)}_{\frac{\partial f}{\partial z}} dz + \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)}_{\frac{\partial f}{\partial \bar{z}}} d\bar{z}$$

which tells us how to define the operators ∂ and $\bar{\partial}$ on $\Omega^0_{S,\mathbb{C}}$. Explicitly, we have $\partial f = \frac{\partial f}{\partial z} dz$ and $\bar{\partial} = \frac{\partial f}{\partial \bar{z}} d\bar{z}$. Likewise, computing the derivative $d : \Omega^1(S, \mathbb{C}) \to \Omega^2(S, \mathbb{C})$ locally gives

$$\begin{aligned} d(f \cdot dz + g \cdot d\bar{z}) &= \bar{\partial}(f \cdot dz) + \partial(g \cdot d\bar{z}) \\ &= \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) dz \wedge d\bar{z}. \end{aligned}$$

§ 4.2 Holomorphic 1-forms. A formal consequence of the definitions is that $\bar{\partial} f = 0$ is locally the Cauchy–Riemann equation, thus the subspace of holomorphic functions in $\Omega^0(S, \mathbb{C})$ is precisely $\ker \bar{\partial}$. In particular for a holomorphic function f, we have $df = \partial f = f' \cdot dz$ locally where f' is the usual complex derivative.

We extend this idea to define holomorphic 1-forms, which will be holomorphic complex linear forms.

Definition 4.3. A *holomorphic 1-form* is a $(1, 0)$-form satisfying $\bar{\partial}\beta$=0.

Locally if $\beta = f \cdot dz$, then β is holomorphic if and only if f is holomorphic.

Note for a holomorphic 1-form β we have $d\beta = \partial\beta + \bar{\partial}\beta = 0$, so holomorphic 1-forms are always closed. This gives an analogue to Cauchy's theorem: if $Q \subset S$ is a surface with boundary, then by Stokes' theorem we have

$$\int_{\partial Q} \beta = 0.$$

Thus the integral of a holomorphic 1-form vanishes on any closed contour. The 'meromorphic' analogue is as follows– let α be a meromorphic 1-form on S, i.e. a 1-form that is holomorphic on the complement of a discrete subset $D \subset S$ and is given locally as $\alpha = f \cdot dz$ for some meromorphic function f (one checks this notion is well-defined independent of chart). Choosing D minimal, the points of D give the poles of α. Let p be a pole, C a small loop around p.

Definition 4.4. The residue of α at p is

$$\mathrm{Res}_p(\alpha) = \frac{1}{2\pi i} \int_C \alpha$$

which is the usual residue in local coordinates (i.e. the coefficient of $z^{-1}dz$ in a Laurent series expansion of α.)

Proposition 4.5. *Let α be a meromorphic 1-form on a compact Riemann surface S. Then the sum of the residues of α over all poles is zero.*

Proof. By compactness, the set of poles is finite say $p_1, ..., p_n$. Around each p_j, choose a small disc D_j with boundary given by a loop C_j. We have

$$\sum_{j=0}^{n} \mathrm{Res}_{p_j}(\alpha) = \sum_{j=0}^{n} \left(\frac{1}{2\pi i} \int_{C_j} \alpha\right) = \frac{1}{2\pi i} \int_{\bigcup_j C_j} \alpha.$$

Set $\tilde{S}$ to be the complement of $D_1 \cup ... \cup D_n$. Then the boundary of $\tilde{S}$ is $-\bigcup_j C_j$ and α is a holomorphic 1-form on $\tilde{S}$, so we have $\int_{\bigcup_j C_j} \alpha = -\int_{\partial\tilde{S}} d\alpha = 0$. □

§ 4.3 The Laplacian. Since $d^2 = (\partial + \bar{\partial})^2 = 0$, we have $\partial\bar{\partial} = -\bar{\partial}\partial$. This gives a canonical operator $\Omega^0_S \to \Omega^2_S$ on a Riemann surface.

Definition 4.6. The *Laplacian* is defined to be $\Delta = 2i \cdot \partial\bar{\partial}$. We say a function f is *harmonic* if $\Delta f = 0$.

Choosing a local complex coordinate $z = x + iy$, we have $\Delta f = -\frac{1}{2}(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}) dx \wedge dy$ so this is just the usual Laplacian (an operator that takes smooth functions to smooth functions) multiplied by a chosen volume-form. The presentation using differential operators removes the choice of this volume form and makes the operator canonical.

If f is holomorphic, then its real and imaginary parts are harmonic. Indeed, writing $\mathrm{Re}(f) = \frac{1}{2}(f + \bar{f})$, we see that $\partial\bar{\partial}(f + \bar{f}) = \partial\bar{\partial} f + \overline{\bar{\partial}\partial f} = 0$, likewise for the imaginary part.

Exercise 4.7. If φ is real valued and defined on a neighbourhood $U \subset S$ of p, show that there is a $U' \subset U$ such that $\varphi|_U'$ is the real part of a harmonic function on U'. Thus every real function has a corresponding imaginary 'harmonic conjugate'.

Exercise 4.8 (Maximum principle). Suppose φ is non-constant, real valued, harmonic on a connected open set $U \subset S$. Then show that for any $x \in U$ there is a point $x' \in U$ with $\varphi(x') > \varphi(x)$.

Theorem 4.9 (Inverting the Laplacian). *Suppose* ρ *is a* 2*-form on a compact surface* S*. Then* $\int_S \rho = 0$ *if and only if there is a smooth function* f *with* $\Delta f = \rho$*, and if so* f *is unique up to the addition of a constant.*

This is the main technical result of the course. We use it to prove the uniformisation theorem and the Riemann-Roch theorem, which in turn can be used to show that compact Riemann surfaces are algebraic.

Remark 4.10. The result on inverting the Laplacian really applies to all elliptic linear partial differential operators on compact manifolds. In this context, the result is often called the 'Fredholm alternative'.

§ 5 Topological applications

Lecture 5, 07/11/23

We will now study the applications of the theory developed– in particular the Euler characteristic, the Riemann–Hurwitz theorem, and Dolbeault cohomology. We will later also look at analytic applications, solving Laplace's equation on a Riemann surface.

§ 5.1 The Euler characteristic. This is a purely topological notion. We will use some techniques from differential topology without proving them, and this will result in the lecture being a little sketchier. The goal is to understand the topological genus of a Riemann surface through differential forms.

Let S be a smooth surface. We can triangulate it to define the Euler characteristic as

$$\chi(S) = \#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\}.$$

The important fact is that this is independent of the triangulation chosen; one proves this by showing that the Euler characteristic is invariant under refinement (subdivision) of the triangulation and that any two triangulations admit a common refinement.

By explicit triangulation, one can calculate that the Euler characteristic of Σ_g (the compact surface of genus g) is $2 - 2g$. In fact this is precisely how we define genus, i.e. for a surface S we have

$$g(S) = 1 - \frac{1}{2}\chi(S).$$

But the genus appears in other forms, for example in the first de Rham cohomology as we have seen previously. We will see how the two notions are related. Suppose S is oriented, and let α be a 1-form such that the set $\Delta \subset S$ where α vanishes is discrete. For any $p \in \Delta$, we can choose local coordinates x_1, x_2 such that $p = (0, 0)$ and $\alpha = \alpha_1 \, dx_1 + \alpha_2 \, dx_2$. Thus for small $r > 0$, the only zero of the function

$$(\alpha_1, \alpha_2) : \overline{B(0, r)} \to \mathbb{R}^2$$

is at the origin. In particular, when restricted to the boundary, $(\alpha_1, \alpha_2)|_{\partial B(0,r)}$ gives a curve in $\mathbb{R}^2 \setminus 0$ which has a well-defined winding number (the signed number of times the curve wraps around the origin anticlockwise). One checks that this is independent of local coordinates (diffeomorphism invariance) and independent of choice of r (homotopy invariance) since it is a continuous integer-valued function. Thus we can make sense of the *winding number of a* 1*-form around a point*, also called the *index of the* 1*-form at the point.* Write $\nu_p(\alpha)$ for this number, noting that it can be computed in various ways for example by integrating $d\theta$ around $(\alpha_1, \alpha_2)\partial B(0, r)$ in local polar coordinates.

Note that this notion makes sense even for $p \notin \Delta$, but if $\alpha(p) \neq 0$ then we can choose r sufficiently small so that $(\alpha_1, \alpha_2) : \overline{B(0, r)} \to \mathbb{R}^2$ does not hit the origin. Thus the winding number of the boundary curve around the origin is necessarily zero.

Proposition 5.1 (Poincaré–Hopf theorem for smooth 1-forms). *For* S, α, Δ *as above, the Euler characteristic of* S *can be computed as*

$$\chi(S) = \sum_{p \in \Delta} \nu_p(\alpha).$$

Very sketchy proof. One could (dually, equivalently) use vector fields rather than 1-forms by fixing a volume form. The result is then the usual Poincaré–Hopf theorem[1].

One first uses the Gauss map to show that the number $\sum_{p\in\Delta}\nu_p(\alpha)$ is indepentent of α, see [Milnor, *"Topology from the differentiable viewpoint"*, Chapter 6]. This lets us choose a specific 1-form whose winding number we can relate to the triangulation. In particular, given a triangle we choose a function f definedon a neighbourhood which has minima on the three vertices, maxima in the interior of the face, and saddle points on the edges. The zeroes of df are the critical points of f, precisely the seven points described above. Then the winding number of df around a minimum or a maximum is $+1$ by an explicit calculation (using Taylor series, say), and at a saddle point it is -1. One glues these functions to get a globally defined smooth function, and use its differential to get the required computation. □

On a compact Riemann surface S, we like holomorphic and meromorphic 1-forms. To a holomorphic 1-form α, we associate a real 1-form $A = \frac{1}{2}(\alpha+\bar{\alpha})$. Locally if $\alpha = f(z)dz$, then $A = \frac{1}{2}(f(z)dz + \bar{f}(z)d\bar{z})$ hence A vanishes if and only if α does. In particular A vanishes on a discrete set Δ so can be used to compute the Euler characteristic of S.

We will show that the index of A at $p \in S$ computes the vanishing multiplicity of α at p (defined by writing $\alpha = f(z)dz$ locally), written $m_p(\alpha)$. This will give the following version of the Poincaré–Hopf formula.

Proposition 5.2 (Poincaré–Hopf theorem for holomorphic 1-forms). *For S a compact Riemann surface with a holomorphic 1-form α vanishing on a discrete set Δ, we have*

$$\sum_{p\in\Delta} m_p(\alpha) = -\chi(S).$$

Proof. Say $f = f_1 dz + if_2 dz$ for smooth real functions f_1, f_2, and local coordinate $z = x+iy$. Then we see that $A = f_1 dx - f_2 dy$, i.e. the winding number of A around $p \in \Delta$ is the negative of the winding number of $(f_1, f_2) = f : \partial B(0,r) \to \mathbb{R}^2$. Now by the argument principle, this number $-\nu_p(A)$ is equal to the vanishing multiplicity of $f(z)$ at p. Thus the total number of zeros of α with multiplicity is $2g-2 = -\chi(S)$. □

Note this quantity has to be positive. Thus if $g = 0$ there are no holomorphic 1-forms at all. If $g = 1$, any holomorphic 1-form would have to be nowhere vanishing (and indeed such 1-forms exist).

Holomorphic 1-forms are too restrictive. Instead we consider meromorphic 1-forms. Fix a volume-form ω on S, locally given by $ig(z)dz\wedge d\bar{z}$ for a positive function g. This gives a hermitian metric on T^*S: for a cotangent vector $v \in T_p^*S$, we define $|v|^2$ so that $v\wedge\bar{v} = |v|^2\omega(p)$. Note $|v|^2 \geqslant 0$, and it vanishes if and only if $v = 0$.

Let α be a meromorphic 1-form on S. Choose a real-valued function ρ on $\mathbb{R}$ with $\rho(t) = 1$ for small t and $\rho(t) = t^{-1}$ for large t. Define a new differential form $\tilde{\alpha}$ satisfying $\tilde{\alpha} = \rho(|\alpha|^2)\alpha$ away from the poles of α, and $\tilde{\alpha} = 0$ at poles of α. Then locally near a pole of α, we have

$$\tilde{\alpha} = \frac{1}{|f(z)|^2} f(z) R\cdot dz = \bar{f}(z)^{-1} R\cdot dz$$

where R is a smooth positive function. In particular this is a smooth 1-form that vanishes at the poles. Thus the zero set of $\tilde{\alpha}$ is the set of zeros and poles of α. Moreover, at a pole of order d of α, $\tilde{\alpha}$ has index $-d$ since $\bar{f}$ is anti-holomorphic. This gives us the following.

Corollary 5.3. *If α is a nontrivial meromorphic 1-form on a compact Riemann surface of genus g, then the number of zeros of α minus the number of poles of α (counted with multiplicity) is $2g-2$.*

§ 5.2 Riemann–Hurwitz formula. The results of the previous section allow for a quick proof of this classical result. Let $f : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. We defined a multiplicity k_x associated to each $x \in X$ such that locally the map is $z \mapsto z^{k_x}$, with $k_x = 1$ at all points except for a finite set of ramification points.

Definition 5.4. The *ramification index* of f is defined to be $R_f = \sum_{x\in X}(k_x - 1)$, noting that the sum is finite.

We also defined the degree d of f, the number of points in any preimage counted with multiplicity. These quantities are related as follows.

Theorem 5.5 (Riemann–Hurwitz). *The genera g_X of X and g_Y of Y are related as*

$$2-2g_Y = d(2-2g_X) - R_f.$$

[1]One of the consequences of the Poincaré–Hopf theorem is the hairy ball theorem.

Proof. We will prove this assuming that there is a meromorphic 1-form on β on Y. This is a harmless assumption, we will later use the Riemann–Roch theorem to prove that this always holds for every Riemann surface.

Given this, $f^*\beta$ is a meromorphic 1-form on X. If $x \in X$ is a point around which f looks like $z \mapsto z^k$, then writing $\beta = g(w)dw$ locally around $f(x)$ gives $f^*\beta = kz^{k-1}g(z^k)dz$. Thus if β has a zero of order ℓ at y=f(x), then $f^*\beta$ has a zero of order $k\ell + k - 1$ at x. So the contribution to the count of zeros and poles of $f^*\beta$ coming from y is

$$\sum_{x\in f^{-1}(y)} (k_x\ell + k_x - 1) = d \cdot \ell + \sum_{x\in f^{-1}y} (k_x - 1).$$

The result follows from the formula of Euler characteristic in terms of meromorphic 1-forms. ☐

Exercise 5.6. Another application is the 'degree–genus formula', which states that given a surface $S \subset \mathbb{P}^2$ defined using a homogeneous degree d polynomial, we have $g(S) = \frac{1}{2}(d-1)(d-2)$. As a corollary, one notes that not all Riemann surfaces can be embedded in $\mathbb{P}^2$.

§ 5.3 Dolbeault cohomology. We continue to relate complex differential forms to geometry. Recall that we used the exterior derivative $d : \Omega^q(S) \to \Omega^{q+1}(S)$ to define the de Rham cohomology.We now use the operator $\bar\partial : \Omega^{p,q}(S) \to \Omega^{p,q+1}(S)$ to define *Dolbeault cohomology*, where $\Omega^{p,q}(S) \subset \Omega^{p+q}(S)$ is the space of (p,q)-forms on S. For example, $\Omega^{0,0}(S)$ is the space of complex functions on S, while $\Omega^{0,1}(S)$ is the space of 1-forms locally given by $f d\bar z$ for a complex function f.

Definition 5.7. The Dolbeault cohomology groups of S are

$$\begin{aligned}
H^{0,0}(S) &= \ker(\bar\partial : \Omega^{0,0}(S) \to \Omega^{0,1}(S)) \\
&= \{\text{global holomorphic functions}\} \\
H^{1,0}(S) &= \ker(\bar\partial : \Omega^{1,0}(S) \to \Omega^{1,1}(S)) \\
&= \{\text{global holomorphic 1-forms}\} \\
H^{0,1}(S) &= \frac{\Omega^{0,1}(S)}{\operatorname{im}(\bar\partial : \Omega^{0,0}(S) \to \Omega^{0,1}(S))} \\
H^{1,1}(S) &= \frac{\Omega^{1,1}(S)}{\operatorname{im}(\bar\partial : \Omega^{1,0}(S) \to \Omega^{1,1}(S))}.
\end{aligned}$$

Exercise 5.8. For $S = \mathbb{C}$, note that $H^1(S) = 0$ but $H^{1,0}(S)$ is infinite dimensional. Now consider a $(0,1)$-form on $\mathbb{C}$ of the form $f d\bar z$. Show that there is a complex function g on $\mathbb{C}$ with $\bar\partial g = f d\bar z$. So $H^{0,1}(\mathbb{C}) = 0$. This is called the $\bar\partial$–Poincaré lemma.

Note that while we have good geometric interpretations of $H^{0,0}$ and $H^{1,0}$, interpreting $H^{0,1}$ takes a bit more work. Suppose we try to find a meromorphic function on a Riemann surface S with a simple pole at $p \in S$ and no other poles. This would, for instance, show that $S \cong \mathbb{P}^1$. Finding such a function locally is trivial– just pick a local coordinate z in a neighbourhood $U \ni p$ and choose the function $\frac{1}{z}$. Let β be a smooth cut-off function near p (i.e. a real function supported on U and equal to 1 near p). So $\beta(\frac{1}{z})$ induces a smooth function on $S \setminus \{p\}$. We want a smooth function g on S such that $g + \beta(\frac{1}{z})$ is holomorphic on $S \setminus \{p\}$. Now $A = \bar\partial\beta(\frac{1}{z})$ has compact support in $U \setminus \{p\}$ (since $\beta = 1$ near p), and we can extend A to a $(0,1)$-form on S by extending by zero over p. So we want to solve $\bar\partial g = -A$ for $A \in \Omega^{0,1}(S)$, and a solution exists if and only if the class $[A] \in H^{0,1}(S)$ equals zero.

Corollary 5.9. *Suppose S is a compact Riemann surface with $H^{0,1}(S) = 0$. Then $S \cong \mathbb{P}^1$.*

Theorem 5.10. *Let S be a compact Riemann surface. We have the following operations and compatibilities on Dolbeault cohomology.*

1. *The map $\sigma : H^{1,0}(S) \to H^{0,1}(S)$, taking a global holomorphic one-form α to its conjugate $\bar\alpha$, is a complex anti-linear isomorphism.*
2. *The bilinear map $B : H^{1,0}(S) \times H^{0,1}(S) \to \mathbb{C}$ given by $B(\alpha, [\theta]) = \int_S \alpha \wedge \theta$ is a perfect pairing. In particular there is an isomorphism $H^{0,1}(S) \cong (H^{1,0}(S))^*$.*
3. *The map $H^{1,0}(S) \oplus H^{0,1}(S) \to H^1(S)$ defined by $(\alpha, \theta) \mapsto \iota(\alpha) + \overline{\iota(\sigma^{-1}\theta)}$ is an isomorphism, where $\iota : H^{1,0}(S) \to H^1(S)$ sends a holomorphic 1-form to its cohomology class.*
4. *The map $\gamma : H^{1,1}(S) \to H^2(S)$ defined by the natural inclusion $\operatorname{im}(\bar\partial : \Omega^{0,1} \to \Omega^2) \subset \operatorname{im}(d : \Omega^1 \to \Omega^2)$ is an isomorphism.*

Proof. These are all consequences of inverting the Laplacian. We first show σ is surjective. Given $[\theta] \in H^{0,1}(S)$ we want a $\theta' = \theta + \bar\partial f$ with $\partial\theta' = 0$, since then $\alpha = \overline{\theta'}$ is a holomorphic 1-form and $[\theta] = \sigma(\alpha)$. So we want to solve $\partial\bar\partial f = -\partial\theta$. But since $\int_S \partial\theta = 0$, we can invert the Laplacian $\partial\bar\partial = \frac{1}{2}i\Delta$ to produce f.

The composition of σ with B is, up to a factor, the Hermitian pairing $\langle\alpha,\beta\rangle = \int_S \alpha \wedge \bar\beta$ which is positive definite. So B is a perfect pairing and σ is injective.

The remaining two can be proved similarly, and are left as exercises. □

We can now relate the topological genus to complex geometry.

Corollary 5.11. *For a compact Riemann surface S of genus g, we have* $\dim H^{1,0}(S) = \dim H^{0,1}(S) = g$.

Lecture 6, 14/11/23

§ 6 Inverting the Laplacian

Having seen the utility of theorem 4.9, we will now repay the technical debt and look at its proof. This will invoke key results from functional analysis which we will recall as necessary.

§ 6.1 The Dirichlet norm. Let S be a Riemann surface, $\alpha \in \Omega^{1,0}(S)$ given locally by $f dz$ for f a smooth $\mathbb{C}$-valued function. Then $i\alpha \wedge \bar\alpha$ is a $(1,1)$-form on S given locally as $i \cdot |f|^2 dz \wedge d\bar z = |f|^2 dx \wedge dy$. In particular this is a non-negative real 2-form hence we have

$$\|\alpha\|^2 := \int_S i\alpha \wedge \bar\alpha \geqslant 0.$$

When α has compact support, this quantity is finite and thus defines a norm on the space of compactly supported $(1,0)$-forms $\Omega^{1,0}_C(S)$. This norm is induced by the hermitian inner product $\langle\alpha,\beta\rangle = \int_S i\alpha\wedge\bar\beta$.

Fixing a volume-form (i.e. a nowhere vanishing real 2-form) ω, we can define a ***pointwise norm*** $|\alpha|^2 \in \Omega^0(S)$, characterised by the property $i\alpha \wedge \bar\alpha = |\alpha|^2\omega$. Then we have $\|\alpha\|^2 = \int_S |\alpha|^2\omega$, i.e. the norm on $\Omega^{1,0}_C(S,\mathbb{C})$ comes from the L^2-norm on $\Omega^0_C(S,\mathbb{R})$. The norm on $(1,0)$-forms is on the other hand independent of the choice of volume-form.

Recall there is an isomorphism $\Omega^{1,0}(S) \cong \Omega^{0,1}(S)$ given by complex conjugation, and there is an orthogonal decomposition $\Omega^1(S,\mathbb{C}) = \Omega^{1,0}(S) \oplus \Omega^{0,1}(S)$. Writing $\alpha^{1,0}$ and $\alpha^{0,1}$ for the two components of a complex 1-form α with respect to this decomposition, we can naturally extend the norm above to $\Omega^1_C(S,\mathbb{C})$ as $\|\alpha\|^2 = \|\alpha^{1,0}\|^2 + \|\alpha^{0,1}\|^2$. Here $\|\alpha^{0,1}\|$ is by definition $\|\overline{\alpha^{0,1}}\|$.

The space of compactly supported real 1-forms is naturally the subspace $\{\alpha \in \Omega^1_C(S,\mathbb{C}) \mid \alpha^{1,0} = \overline{\alpha^{0,1}}\}$. Thus the norm restricts, and $\Omega^1_C(S,\mathbb{R})$ has an inner product and a norm given as

$$\langle A, B\rangle = 2i\int_S A^{1,0} \wedge B^{0,1}, \qquad \|A\|^2 = 2\|A^{1,0}\|^2.$$

We can then use the exterior derivative to induce a norm on the space of smooth real-valued functions.

Definition 6.1. The *Dirichlet inner product* on $\Omega^0_C(S,\mathbb{R})$ is given by $\langle f, g\rangle_D := \langle df, dg\rangle$. This induces the *Dirichlet norm*, given by $\|f\|_D = \|df\|$. These definitions can also be made for the complex vector space $\Omega^0(S,\mathbb{C})$ analogously.

Note that the Dirichlet inner product (norm) is <u>not</u> an inner product (resp. norm) since, for instance, any non-zero constant function f satisfies $\|f\|_D = 0$. Conversely if $\|f\|_D = \|df\|$ vanishes, then $df = 0$ since $\|\cdot\|$ is a norm on $\Omega^1_C(S,\mathbb{C})$ and hence f is constant. Thus it makes sense to consider the space $\Omega^0_C(S,\mathbb{R})/\mathbb{R}$ where we quotient out the subspace of constant functions.

Proposition 6.2. *The Dirichlet inner product (norm) is an inner product (resp. norm) on $\Omega^0_C(S,\mathbb{R})/\mathbb{R}$ and turns it into a pre-Hilbert space (i.e. a vector space equipped with an inner product).*

We remark that the definition of the Dirichlet inner product makes sense even when just one of the arguments has compact support.

Lemma 6.3. *If f, g are smooth functions on S and f has compact support, then* $\langle f, g\rangle_D = \int_S f \cdot \Delta g = \int_S \Delta f \cdot g$.

Proof. A partition of unity argument reduces this to the local case. We then have $\langle f, g\rangle_D = 2i\int_S \partial f \wedge \bar\partial g$, so writing $\partial f \wedge \bar\partial g = \partial(f\bar\partial g) - f \cdot \partial\bar\partial g$ and using Stokes' theorem gives the required result. Note this is just integration by parts. □

We can now prove one direction of theorem 4.9.

Corollary 6.4. *Let S be a compact Riemann surface with a real 2-form ρ. Then up to adding a constant there is at most one $f \in \Omega^0(S, \mathbb{C})$ satisfying $\Delta f = \rho$, and such an f exists only if $\int_S \rho = 0$.*

Proof. To show uniqueness of the solution, we show $\Delta f = 0$ (if and) only if f is a constant. Indeed we have $\int_S f\Delta f = ||f||_D^2$, which vanishes if and only if $df = 0$ (i.e. f is constant) by properties of norms.

If $\rho = \Delta f$ for some function f, then $\int_S \rho = 2i\int_S \partial\bar{\partial} f = 2i\int_S d(\partial f)$ which vanishes by Stokes' theorem. □

§ 6.2 The heart of the proof. Our goal is to show that a real 2-form ρ on a compact Riemann surface S that satisfies $\int_S \rho = 0$ is in the image of Δ. Suppose $\int_S \rho = 0$, and consider the pre-Hilbert space $\Omega^0(S, \mathbb{R})/\mathbb{R}$.

Exercise 6.5. For $f \in \Omega^0(S, \mathbb{R})$, we have $\Delta f = \rho$ if and only if for all $g \in \Omega^0(S, \mathbb{R})$, $\int_S g(\rho - \Delta f) = 0$.

Now we know for all f, g in this space, we have $\int_S \psi(\rho - \Delta\varphi) = \int_S \psi\rho - \langle\varphi, \psi\rangle_D$. In particular, $\Delta f = \rho$ if and only if $\int_S g\rho = \langle f, g\rangle_D$ for all g. This reduces the problem to studying the linear operator

$$\hat{\rho} : \Omega^0(S, \mathbb{R})/\mathbb{R} \to \mathbb{R}, \qquad g \mapsto \int_S g\rho$$

which is well-defined since $\int_S \rho = 0$. Claim this is a bounded linear operator on $\Omega^0(S, \mathbb{R})/\mathbb{R}$ and hence extends to the metric completion $H = (\Omega^0(S, \mathbb{R})/\mathbb{R})^{\wedge}$ which is the Hilbert space whose points are given by equivalence classes of Cauchy sequences in $\Omega^0(S, \mathbb{R})/\mathbb{R}$.

Then the Riesz representation theorem states that every bounded linear operator on H is of the form $\langle\cdot, f\rangle_D$ for some $f \in H$, thus giving a weak solution to the problem– we have an $f \in H$ such that $\hat{\rho}(g) = \langle f, g\rangle_D$ for all g. Finally, we claim that f is in fact in $\Omega^0(S, \mathbb{R})/\mathbb{R}$, i.e. f is smooth. This finishes the proof of theorem 4.9.

§ 6.3 The boundedness claim. We first prove an analytic result about smooth functions on $\mathbb{R}^2$. Let $U \subset \mathbb{R}^2$ be a bounded convex open set of diameter d and area A with respect to the standard Lebesgue measure $d\mu$. For a smooth function φ defined on an open set containing $\bar{U}$, write $\bar{\varphi} = \frac{1}{A}\int_U \varphi d\mu$ for its average. We show the deviation of φ from its average is bounded.

Lemma 6.6 (Poincaré inequality). *For U, d, A, φ as above and $x \in U$, we have*

$$|\varphi(x) - \bar{\varphi}| \leqslant \frac{d^2}{2A}\int_{y\in U} \frac{|\nabla\varphi(y)|}{|x-y|} d\mu.$$

Proof. Up to translation and adding a constant, we can assume $x = \mathbf{0}$ is the origin and $\varphi(\mathbf{0}) = 0$ so we have to bound $|\bar{\varphi}|$. Since U is convex, we can use polar coordinates to write $\bar{\varphi} = \frac{1}{A}\int_0^{2\pi}\int_0^{R(\theta)} \varphi(r, \theta) r dr d\theta$. Moreover, we can write $\varphi(r, \theta) = \int_0^r \frac{\partial\varphi}{\partial s}(s, \theta) ds$ since $\varphi(\mathbf{0}) = 0$. Thus we have

$$\begin{aligned}
\bar{\varphi} &= \frac{1}{A}\int_{\theta=0}^{2\pi}\int_{r=0}^{R(\theta)}\int_{s=0}^{r} \frac{\partial\varphi}{\partial s}(s, \theta) r ds dr d\theta \\
&= \frac{1}{A}\int_{\theta=0}^{2\pi}\int_{s=0}^{R(\theta)} \left(\int_{r=s}^{R(\theta)} r dr\right) \frac{\partial\varphi}{\partial s}(s, \theta) ds d\theta
\end{aligned}$$

and $\int_{r=s}^{R(\theta)} r dr = \frac{1}{2}(R(\theta)^2 - s^2) \leqslant \frac{1}{2}R(\theta)^2 \leqslant \frac{1}{2}d^2$. Additionally, we know $|\frac{\partial\psi}{\partial s}| \leqslant |\nabla\varphi|$ so putting these together, we have the required inequality

$$\begin{aligned}
|\bar{\varphi}| &\leqslant \frac{d^2}{2A}\int_{\theta=0}^{2\pi}\int_{s=0}^{R(\theta)} |\nabla\varphi(s, \theta)| \cdot \frac{1}{s} s ds d\theta \\
&\leqslant \frac{d^2}{2A}\int_{y\in U} \frac{|\nabla\varphi(y)|}{|y|} d\mu.
\end{aligned}$$

□

Corollary 6.7. *Under the hypotheses above, we have*

$$\int_U |\varphi - \bar{\varphi}|^2 d\mu \leqslant \frac{d^2\pi}{2A}\int_U |\nabla\varphi|^2 d\mu.$$

Proof. This is essentially a consequence of Yang's inequality on convolutions– for functions g, h on $\mathbb{R}^2$ such that g is integrable and h is square-integrable, we have

$$||g * h||_{L^2} \leqslant ||g||_{L^1} \cdot ||h||_{L^2}$$

where $g * h(x) = \int_{y \in \mathbb{R}^2} g(y)h(x-y)d\mu$ is the ***convolution*** of g and h, and $||\cdot||_{L^1}, ||\cdot||_{L^2}$ denote the L^1 and L^2 norms respectively. Convolution $*$ is a commutative and associative operation whenever defined.

To prove the corollary we choose $g(x) = \frac{d^2}{2A}\frac{1}{|x|}$ if $0 < |x| < d$ and 0 otherwise, noting this is discontinuous but nonetheless integrable with $||g||_{L^1} = \frac{d^3\pi}{A}$. Likewise, choose $h(x) = |\nabla\varphi(x)|^2$ for $x \in U$ and 0 otherwise. Then $g * h$ is a positive function on $\mathbb{R}^2$ and the Poincaré inequality says $|\varphi(x) - \bar{\varphi}| \leqslant |g * h(x)|$ for all $x \in U$. In particular $\int_U |\varphi - \bar{\varphi}|^2 d\mu \leqslant ||g * h||_{L^2}^2$, so Yang's inequality gives the result. □

We can now show $\hat{\rho} : \Omega^0(S, \mathbb{R})/\mathbb{R} \to \mathbb{R}$ given by $g \mapsto \int_S g\rho$ is a bounded operator. We first reduce to the case when ρ is supported in a single coordinate chart– since integration over S defines an isomorphism $H^2(S) \cong \mathbb{R}$, we know $[\rho] = \int_S \rho = 0$ i.e. $\rho = d\theta$ for some 1-form. By compactness of S, we can use a partition of unity argument to write $\theta = \theta_1 + ... + \theta_n$ where each θ_i is a 1-form supported on a bounded convex coordinate chart. Then each $\rho_i = d\theta_i$ is a 2-form supported on such a chart, satisfies $\int_S \rho_i = 0$, and we have $\rho = \rho_1 + ... + \rho_n$. Thus $\hat{\rho} = \hat{\rho}_1 + ... + \hat{\rho}_n$, and it suffices to show each operator $\hat{\rho}_i$ is bounded.

Assume ρ is a 2-form supported on a bounded convex coordinate chart $U \subset \mathbb{R}^2$. Using the standard nowhere-vanishing volume form $dx \wedge dy$, we can identify 2-forms with functions so that $\rho = h dx \wedge dy$ for a smooth function h supported on U. A function $\varphi \in \Omega^0(S)$ induces a function on U by restriction, and we have

$$|\hat{\rho}(\varphi)| = \left| \int_U h\varphi d\mu \right| = \left| \int_U h(\varphi - \bar{\varphi}) d\mu \right| \leqslant ||h||_{L^2(U)} \cdot ||\varphi - \bar{\varphi}||_{L^2(U)}$$

where we recall $\int_U h\bar{\varphi} d\mu = \bar{\varphi} \int_U h d\mu = 0$ and use the Cauchy–Schwartz inequality. Thus the corollary gives us

$$|\hat{\rho}(\varphi)| \leqslant \frac{d^2\pi}{2A} \cdot ||h||_{L^2(U)} \cdot ||\nabla\varphi||_{L^2(U)}.$$

Lastly, note that $||\nabla\varphi||_{L^2(U)} \leqslant ||d\varphi||_{L^2(X)} = ||\varphi||_D$ and write $C = \frac{d^2\pi}{2A} \cdot ||h||_{L^2(U)}$ so that we have the required inequality

$$|\hat{\rho}(\varphi)| \leqslant C \cdot ||\varphi||_D$$

§6.4 The regularity claim. Riesz's representation theorem gives us a sequence of functions $f_1, f_2, ...$ that is Cauchy with respect to the Dirichlet norm, such that for any $g \in \Omega^0(S, \mathbb{R})$ we have $\lim_{i \to \infty} \langle f_i, g \rangle_D = \hat{\rho}(g)$.

We first show that the sequence (f_i) converges to an L^2 function on S. This is easy locally– on any coordinate chart U we may assume f_i has integral zero by adding a constant, so that we have $||f_i - f_j||_{L^2(U)} \leqslant C \cdot ||f_i - f_j||_D$ by corollary 6.7. Thus the sequence is Cauchy with respect to the L^2 norm which is complete on the space of square-integrable functions.

To show the sequence is Cauchy in $L^2(S)$, suppose U, V are open coordinate charts with $U \cap V \neq \emptyset$ and (f_i) converges in $L^2(U)$. We know there are constants $c_1, c_2, ...$ such that $(f_i + c_i)$ converges in $L^2(V)$, and up to adding a fixed constant to each c_i, we may also assume the two limits agree on $L^2(U \cap V)$. But this implies $c_i \to 0$ so in fact we could have chosen $c_i = 0$ to begin with. Since S is connected, it follows that the sequence (f_i) converges in the L^2 norm locally on each chart, and hence converges to some $f \in L^2(S)$.

It remains to show f is smooth, which can be done locally. This is then the content of the following result.

Proposition 6.8 (Weyl's lemma). *Suppose U is a bounded open set in $\mathbb{C}$ and $\rho \in \Omega^2(U)$ is a smooth 2-form. If $f \in L^2(U)$ is such that for any smooth compactly supported function g we have $\int_U (\Delta f)g = \int_U g\rho$ then f is smooth and satisfies $\Delta f = \rho$.*

Proof. Omitted, see [Don04]. The first step is to reduce to $\rho = 0$ by convolutions, so that we have a weakly harmonic function f (i.e. an L^2 function such that $\Delta f = 0$ in a weak sense). Then we want to show that weakly harmonic functions are smooth, which we do by finding an explicit inverse to the Laplace operator in terms of convolutions. □

Lecture 7, 21/11/23

§7 The Riemann–Roch theorem

Let S be a compact Riemann surface of genus g, and let $D = \{p_1, \dots, p_d\}$ be a set of distinct points in S.

Definition 7.1. We denote by $H^0(D)$ the vector space of meromorphic functions on S with at worst a simple pole at each p_j. Write $H^0(K - D)$ for the vector space of holomorphic 1-forms vanishing at each p_j.

The cohomological notation will be explained later. For now, these are complex vector spaces keeping track of functions and 1-forms with interesting properties and the Riemann–Roch theorem will relate their dimensions. A partial result in the form of an inequality was first proven by Riemann, and Roch subsequently completed the work by computing the error term. This, together with its many generalisations (the Hirzebruch–Riemann–Roch theorem, the Grothendieck–Riemann–Roch theorem, and the Atiyah–Singer index theorem) is one of the most fundamental results of algebraic geometry.

Theorem 7.2 (Riemann–Roch). *For S, D as above, we have $\dim H^0(D) - \dim H^0(K - D) = d - g + 1$.*

This is typically shown using algebro-geometric methods (Serre duality), but we don't know that S has an algebraic structure yet. In fact we will use this result to show all compact Riemann surfaces are algebraic, so our proof of the Riemann–Roch theorem will be analytic.

§7.1 $(0,1)$–cohomology classes control meromorphic functions. To motivate the proof, we revisit a construction from before. Suppose we want to construct a meromorphic function f on S with exactly one pole, simple at $p \in S$. We can easily find a smooth function $\psi \in \Omega^0(S \setminus \{p\}, \mathbb{C})$ that is meromorphic near p with a simple pole; for instance by taking the function $\frac{1}{z}$ in a local chart and extending globally by multiplying with a cutoff function β which is 1 near p and vanishes outside a neighbourhood of p. The problem is thus reduced to finding a smooth function $g \in \Omega^0(S, \mathbb{C})$ such that $g - \psi$ is holomorphic on $S \setminus \{p\}$, or equivalently $\bar\partial g = \bar\partial \psi$ on $S \setminus \{p\}$.

Consequently consider the smooth $(0,1)$-form $A = \bar\partial\psi$, which vanishes near p (since ψ is holomorphic there) so extends by 0 across p to give a global form on S. We want to find $g \in \Omega^0(S, \mathbb{C})$ which solves $\bar\partial g + A = 0$, the obstruction to this is precisely the class $[A] \in H^{0,1}(S)$ and we have a solution if and only if $[A] = 0$. But even if $[A] \neq 0$, this obstruction class is defined (up to scaling) independently of choice of ψ. Indeed if $\varphi \in \Omega^0(S \setminus \{p\}, \mathbb{C})$ is meromorphic near p with a simple pole, then for some $\lambda \in \mathbb{C}$ (given by the residue) we have that $\varphi - \lambda\psi$ is smooth everywhere (including near p) so $[\bar\partial\varphi] = \lambda[\bar\partial\psi] \in H^{0,1}(S)$. Thus the obstruction to finding a global meromorphic function is well-defined (up to scaling) as a $(0,1)$-Dolbeault cohomology class. Moreover we can represent this class by a $(0,1)$–form $\bar\partial\psi$ for ψ supported on a small punctured neighbourhood of p, smooth away from p.

For multiple distinct points $p_1, \dots, p_d \in S$, we get $(0,1)$–forms $A_j = [\bar\partial\psi_j]$ supported near p_j from the above construction. Similar arguments show we can find a meromorphic function with simple poles precisely on the p_i provided there are $\lambda_j \in \mathbb{C} \setminus 0$ with $\lambda_1[A_1] + \dots + \lambda_d[A_d] = 0$ in $H^{0,1}(S)$. Indeed, in that case $\psi = \sum_j \lambda_j\psi_j \in \Omega^0(S \setminus \{p_1, \dots, p_d\}, \mathbb{C})$ is non-zero and meromorphic with a simple pole near each p_j, and $\bar\partial\psi$ extends to a global $(0,1)$–form that is cohomologically trivial. Thus there is a smooth function g with $\bar\partial\psi = \bar\partial g$ on $S \setminus \{p_1, \dots, p_d\}$ and $\bar\partial g(p_j) = 0$, so that $g - \psi$ is the required meromorphic function.

We can immediately use this reasoning to prove some results.

Corollary 7.3. *Suppose S has genus g. Then given $g + 1$ points $p_1, \dots, p_{g+1}$ in S, there is a meromorphic function on S with at worst simple poles, at least one pole, and smooth away from $p_1, \dots, p_{g+1}$.*

Proof. We know $H^{0,1}(S)$ has dimension g, so any $g + 1$ elements of $H^{0,1}(S)$ must be linearly dependent. □

In particular, we have a classification of compact Riemann surfaces of genus 0.

Corollary 7.4. *Suppose S has genus 0. Then S is biholomorphic to the Riemann sphere.*

Proof. Indeed then $H^{0,1}(S) = 0$, so we can find a meromorphic function with a simple pole which gives the required biholomorphism. □

§7.2 The residue map. The second important ingredient in proving the Riemann–Roch theorem is the concept of residues. What is the residue of a meromorphic function f at a point $p \in S$? In a local coordinate z, if $f(z) = \sum_i a_i z^i$ then the residue is a_{-1}. With respect to another local coordinate $w = \lambda_1 z + \lambda_2 z^2 + \dots$, the residue is $\lambda_1 a_{-1}$. Noting that $\frac{\partial}{\partial z} = \lambda_1 \frac{\partial}{\partial w} \in T_p^{\mathrm{hol}} S$ (the holomorphic tangent space[2]), we thus see that the quantity $a_{-1} \frac{\partial}{\partial z} \in T_p^{\mathrm{hol}} S$ is independent of coordinates, and this is what we call the *residue* of f at p.

§7.3 The obstruction-class map. For similar reasons, we can also define the obstruction-class at $p \in S$ from the previous section as a map $A : T_p^{\mathrm{hol}} S \to H^{0,1}(S)$ by first fixing a smooth cutoff function β, and then mapping $\frac{\partial}{\partial z} \mapsto [\bar{\partial}\beta \cdot \frac{1}{z}]$. Indeed for a different local coordinate $w = \lambda_1 z + \lambda_2 z^2 + \ldots$, we see that $\frac{1}{z} - \lambda_1 \frac{1}{w}$ is holomorphic near p so $[\bar{\partial}\beta \cdot \frac{1}{z}] = \lambda_1 [\bar{\partial}\beta \cdot \frac{1}{w}]$.

We examine the dual map $A^{\mathsf{T}} : H^{0,1}(S)^* \to (T_p^{\mathrm{hol}} S)^*$. Recall we can identify $H^{1,0}(S)$ with $H^{0,1}(S)^*$ via the perfect pairing $(\alpha, \beta) \mapsto \int_S \alpha \wedge \beta$. Note also that $(T_p^{\mathrm{hol}} S)^*$ is simply the holomorphic cotangent space $T_p^{1,0} S$. But there is an obvious evaluation map $\mathrm{ev} : H^{1,0}(S) \to T_p^{1,0} S$ sending a holomorphic 1-form to its value at p, and we now show that A^{T} is essentially this.

[2]The holomorphic tangent space can be defined as follows. Recall that the tangent space $T_p S$ at $p \in S$ has a natural complex structure $J : T_p S \to T_p S$, which extends to a $\mathbb{C}$-linear complex structure $J : T_p S \otimes \mathbb{C} \to T_p S \otimes \mathbb{C}$ by writing $J(v + iw) = J(v) + iJ(w)$ for $v, w \in T_p S$. Then $J^2 = -\mathrm{Id}$ so J has eigenvalues $\pm i$, and the holomorphic tangent space is the $+i$–eigenspace. Equivalently, one can check that this is precisely the subspace of $T_p S \otimes \mathbb{C}$ where all anti-linear cotangent-vectors (i.e. elements of $T_p^{0,1} S$ vanish.) In local coordinates, the holomorphic tangent space has vectors of the form $\frac{\partial}{\partial z}$ for $z = x + iy$ a local holomorphic coordinate.

Lemma 7.5. *Under the identifications above,* $A^T = 2\pi i \cdot \mathrm{ev}$.

Proof. This is a local result, so choose a coordinate z. Given a $(1,0)$-form α, we see that $A^T(\alpha)$ sends the tangent vector $\frac{\partial}{\partial z}$ to $\langle \alpha, A\frac{\partial}{\partial z}\rangle = \int_S \alpha \wedge (\bar\partial\beta \cdot \frac{1}{z})$. But if α is locally $g(z)dz$, we see that the integral becomes $\int_{\mathbb{C}} \bar\partial\beta \cdot \frac{g(z)}{z} dz$ (noting that β is supported on the coordinate chart around p). Choose a circle γ around p on which $\beta = 1$, so that Stokes' theorem gives us

$$\int_{\mathbb{C}} \bar\partial\beta \cdot \frac{g(z)}{z} dz = \int_\gamma \frac{g(z)}{z} dz.$$

By Cauchy's residue theorem, this is $2\pi i \cdot g(p)$ as required. □

We can now prove the Riemann–Roch theorem. All (co)-tangent spaces considered in the rest of this section will be holomorphic (hence of $\mathbb{C}$-dimension one) and we will simply write T_pS instead of $T_p^{\mathrm{hol}}S$ and likewise T_p^*S instead of $T_p^{1,0}S$.

Proof of the Riemann–Roch theorem. For a collection of points $D = \{p_1, ..., p_d\} \subset S$, we have a residue map $R : H^0(D) \to \bigoplus_{j=1}^d T_{p_j}S$ defined in the obvious way. The kernel of this map is the space of global holomorphic functions, which are all constant since S is compact.

Likewise, we have a map $A : \bigoplus_{j=1}^d T_{p_j}S \to H^{0,1}(S)$ coming from obstruction-classes. We showed that $A(\nu_1, ..., \nu_d) = 0$ if and only if $(\nu_1, ..., \nu_d)$ are the residues of a meromorphic function in $H^0(D)$, i.e. that the kernel of A is precisely the image of R. This gives us an exact sequence

$$0 \to \mathbb{C} \to H^0(D) \xrightarrow{R} \bigoplus_{j=1}^{d} T_{p_j}S \xrightarrow{A} H^{0,1}(S).$$

The proof is now linear algebra– we have $\dim \mathrm{im}(A) + \dim H^0(D) = d + 1$ from the exact sequence, and $\dim \mathrm{im}(A) = \dim H^{0,1}(S) - \dim \ker A^T$. Lastly, note that $A^T : H^{1,0}(S) \to \bigoplus_{j=1}^d T^*_{p_j}S$ is the evaluation map, so its kernel is precisely $H^0(K - D)$. The result follows. □

§ 8 The uniformisation theorem

Theorem 8.1 (The uniformisation theorem). *Let* S *be a non-compact simply connected Riemann surface. Then* S *is biholomorphic to either* $\mathbb{C}$ *or the upper half plane* $\mathbb{H}$.

Note we have already shown any simply connected compact Riemann surface must be biholomorphic to $\mathbb{P}^1$. Since the universal cover of any Riemann surface is a simply connected one, we have the following classification result.

Corollary 8.2. *Any Riemann surface is biholomorphic to one of the following.*

(i) The Riemann sphere $\mathbb{P}^1$*;*

(ii) the complex plane $\mathbb{C}$*;*

(iii) the punctured plane $\mathbb{C} \setminus \{0\} \cong \mathbb{C}/\mathbb{Z}$*;*

(iv) a quotient $\mathbb{C}/\Lambda$ *where* $\Lambda \subset \mathbb{C}$ *is a lattice; or*

(v) a quotient $\mathbb{H}/\Gamma$ *where* $\Gamma \subset \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Aut}\mathbb{H}$ *is a discrete group acting freely on* $\mathbb{H}$.

Proof. Indeed, any Riemann surface is a quotient of its universal cover by an action of its fundamental group and the uniformisation theorem which surfaces can arise as universal covers. Examining their automorphism groups gives the result. □

To prove the uniformisation theorem, we need the following analytic result which allows us to invert Laplacians on S (this time non-compact).

Theorem 8.3. *Let* S *be a connected, simply connected, non-compact Riemann surface. If* ρ *is a real 2-form of compact support on* S *with* $\int_S \rho = 0$*, then there is a real function* φ *on* S *with* $\Delta\varphi = \rho$ *such that* $\varphi \to 0$ *at infinity.*

Here we say $\varphi \to 0$ at infinity in S if for all $\epsilon > 0$ there is a compact $K \subset S$ such that $|\varphi(x)| < \epsilon$ for all $x \notin K$. The proof is omitted, see [Don04]. It is similar to inverting the Laplacian in the compact case, and the simply connectedness provides compensation for the non-compactness.

Proof of the uniformisation theorem. Let S be a non-compact simply connected Riemann surface. As before, we construct a meromorphic function (viewed as a map to $\mathbb{P}^1$) and examine its image.

<u>There is a meromorphic function f with a simple pole.</u> Fix $p \in S$, z a local coordinate around p, and $A = \bar{\partial} \cdot \frac{1}{z}$ a $(0,1)$-form with β a cutoff function. Let $\rho = \partial A$ so that ρ is a complex 2-form with integral zero (Stokes' theorem). Thus by the theorem applied to the real and imaginary parts of ρ separately, we can find g with $\Delta g = \rho$ such that the real and imaginary parts of g tend to 0 at infinity.

Let a be the real part of $A - \bar{\partial} g$. This is a $(0,1)$-form so $\bar{\partial} a = 0$. Since $\partial(A - \bar{\partial} g) = 0$, we have $da = 0$. Thus since $H^1(S) = 0$ (by simply connectedness), there is a real valued φ satisfying $d\varphi = a$. So $d\varphi = (A - \bar{\partial} g) + (\bar{A} - \overline{\bar{\partial} g})$, $A = \bar{\partial} g + \bar{\partial}\varphi$, and $\bar{\partial}(\beta \cdot \frac{1}{z} - (g + \varphi)) = 0$. This gives us the required meromorphic function $f = \beta \cdot \frac{1}{z} - g - \varphi$ with a simple pole at p, such that the imaginary part of f tends to zero at infinity.

<u>The map f is injective.</u> Let $\pm\mathbb{H}$ denote the (open) upper and lower half-planes in $\mathbb{C} \subset \mathbb{P}^1$, and let $S_\pm = f^{-1}(\pm\mathbb{H})$ denote their preimages so that $S_+ \cup S_-$ is a dense open set in S (noting that f is an open map). Note $S_\pm$ are non-empty, since p is a simple pole i.e. f is a local homeomorphism between neighbourhoods of $p \in S$ and $\infty \in \mathbb{P}^1$.

Claim $f_\pm : S_\pm \to \pm\mathbb{H}$ is surjective. Since we know it is an open map, it suffices to show it is also proper. We do this explicitly– if $B \subset \mathbb{H}$ is compact then there is an $\epsilon > 0$ such that $\mathrm{Im}(z) > \epsilon$ for all $z \in B$. Then $f^{-1}(B) = f_+^{-1}(B)$ is compact since $\mathrm{Im} f$ tends to zero at infinity. Similarly f_- is proper.

We can now show $f_\pm : S_\pm \to \pm\mathbb{H}$ is injective. Since f_+ is a holomorphic map between Riemann surfaces, we know it has a degree which is (locally) constant on $\mathbb{H}$. Showing injectivity is equivalent to showing this degree is 1, so suppose f_+ has degree $d_+ \geqslant 2$. By surjectivity of f_+, we see that for all $n \in \mathbb{N}_{\geqslant 0}$ we can find $x_n, \tilde{x}_n \in S_+$ with $f(x_n) = f(\tilde{x}_n) = in$ so either $x_n, \tilde{x}_n$ are distinct or $f'(x_n) = 0$. Since f tends to zero at infinity, the sequences $(x_n), (\tilde{x}_n)$ lie in a compact set K and hence converge to $x, \tilde{x}$ respectively. But the sequence (in) converges to $\infty \in \mathbb{P}^1$, so $f(x) = f(\tilde{x}) = \infty$ and hence $x = \tilde{x} = p$. This contradicts f being a local homeomorphism with non-vanishing derivative on a neighbourhood of p.

So f maps $S_\pm$ bijectively onto the open upper and lower half planes. This allows us to conclude $f : S \to \mathbb{P}^1$ is injective. Indeed if $x_1, x_2 \in S$ are such that $f(x_1) = f(x_2) = z \in \mathbb{P}^1$ then we can find disjoint discs D_1, D_2 around x_1, x_2 and a neighbourhood $N \ni z$ such that $N \subset f(D_1) \cap f(D_2)$. Choose $z' \in N \cap \mathbb{H}$, this gives us distinct $x_1' \in D_1$ and $x_2' \in D_2$ with $f(x_1') = f(x_2') = z'$ contradicting that f is injective on S_+.

<u>The image of f is biholomorphic to $\mathbb{C}$ or $\mathbb{H}$.</u> We have shown $f : S \to \mathbb{P}^1$ is injective and biholomorphic onto its image, which is an open set containing $\mathbb{H} \cup -\mathbb{H} \cup \{\infty\} = \mathbb{P}^1 \setminus \mathbb{R}$. Since S is connected and simply connected, we conclude that $\mathbb{P}^1 \setminus f(S)$ is a compact interval I in $\mathbb{R} \subset \mathbb{P}^1$. If I contains a single point, then $f(S)$ is biholomorphic to $\mathbb{C}$ and if I has nontrivial length, then $f(S)$ is biholomorphic to $\mathbb{H}$ (by writing down an explicit Möbius map). This concludes the proof. ☐

This concludes the main theory of the course, the study of meromorphic functions on Riemann surfaces.

Lecture 8, 28/11/23

§ 9 Line bundles and divisors

The terminology of meromorphic functions is seldom used in modern algebraic geometry. Instead, a generalised theory of line bundles and sections is used. These relate, for example, to the theory of embeddings into projective space.

Let $f_1, \ldots, f_n$ be meromorphic functions on a compact Riemann surface S, and let $\Delta \subset S$ be the (finite) set of poles outside which every f_j is holomorphic. Thus we have a holomorphic map $S \setminus \Delta \to \mathbb{C}^n$, where 'holomorphic' means holomorphic in each variable separately.

Exercise 9.1. Show this can be extended to a holomorphic map from S to $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$. Here we use a natural definition of holomorphic– the standard charts on $\mathbb{P}^n$ cover $\mathbb{C}^n$ with holomorphic transition functions, and a smooth map is holomorphic if it is holomorphic locally.

Instead of understanding this map near Δ, the theory is clearer if we work with line bundles.

Definition 9.2. A rank r holomorphic vector bundle over a Riemann surface S is given by an $r+1$-dimensional complex manifold E with a holomorphic map $p : E \to S$ such that for each $x \in S$, the fiber $p^{-1}(x) = E_x$ has the structure of a complex vector space $\mathbb{C}^r$ and there is an open set $U \subset S$ containing x with a holomorphic and fiberwise linear isomorphism $E|_U \cong U \times \mathbb{C}^n$.

We think of the fibers E_x as 'varying holomorphically with $x \in S$'. We will be interested in line bundles, where the rank r is 1.

There are two further ways to think about vector bundles– first through transition functions. If $\{U_\alpha\}$ is a cover of S which trivialises E, then we get two trivialisations on each intersection $U_\alpha \cap U_\beta$. Namely we have $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^n$ and $E|_{U_\beta} \cong U_\beta \times \mathbb{C}^n$, and on the intersection these differ by a holomorphic endomorphism of $(U_\alpha \times U_\beta) \times \mathbb{C}^n$ that is fiberwise linear. Thus we get a holomorphic map $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$ satisfying $\varphi_{\alpha\alpha} = \mathrm{id}$, $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$, and $\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}$ on triple intersections. Conversely given any cover $\{U_\alpha\}$ and collection of maps $\{\varphi_{\alpha\beta}\}$ satisfying these compatibilities, we get a vector bundle in the natural way.

Another approach to vector bundles is through sheaves. In geometry, we often have assignments such as

$$\{\text{open sets } U \subset S\} \to \mathcal{O}(U) := \{\text{holomorphic functions on } U\}$$

where the right hand side is a vector space, and where restrictions give linear maps $\mathcal{O}(U) \to \mathcal{O}(V)$ whenever $V \subset U$. This is the ***sheaf of holomorphic functions on*** S. Similarly we can define sheaves of continuous functions, smooth functions, differential forms etc.

Definition 9.3. A ***presheaf of abelian groups*** F on S is an assignment of abelian groups $U \mapsto F(U)$ for every open set $U \subset S$, with additive maps $\mathrm{res}_{V,U} : F(U) \to F(V)$ whenever $V \subset U$ satisfying $\mathrm{res}_{U,U} = \mathrm{id}$, $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$ whenever defined. We call $s \in F(U)$ a *section*.

A presheaf is a ***sheaf*** if it satisfies two additional properties. Firstly, if we have a cover $U = \bigcup U_\alpha$ and $s \in F(U)$ is such that $\mathrm{res}_{U_\alpha,U}(s) = 0$ for all α, then $s = 0$. Secondly if we have a cover $U = \bigcup U_\alpha$ and sections $s_\alpha \in F(U_\alpha)$ satisfying $\mathrm{res}_{U_\alpha \cap U_\beta, U_\alpha}(s_\alpha) = \mathrm{res}_{U_\alpha \cap U_\beta, U_\beta}(s_\beta)$, then there is a section $s \in F(U)$ satisfying $\mathrm{res}_{U_\alpha,U}(s) = s_\alpha$.

Example 9.4. A section of a vector bundle $p : E \to S$ over $U \subset S$ is a holomorphic map $s : U \to E|_U$ satisfying $p \circ s = \mathrm{id}$. This gives a sheaf of sections of E, which we denote by $\mathcal{O}(E)$.

The sheaf of holomorphic functions, denoted $\mathcal{O}$, is naturally the sheaf of sections of the ***trivial line bundle*** $\mathbb{C} \times S \to S$.

Example 9.5. One can form the tangent bundle (written $-K$ or T_S) has fibers given by the holomorphic tangent spaces. Likewise, the cotangent bundle K (or T_S^*) has fibers given by the cotangent spaces, and this is the bundle dual to T_S. Sections of T_S and T_S^* are holomorphic vector fields and 1-forms respectively.

§9.1 The Picard group. A line bundle L on S naturally has a *dual* $L^\vee$. This is most easily defined from the transition functions perspective, where the transition functions for $L^\vee$ are inverse pointwise to those of L. Likewise we can take tensor products of line bundles by multiplying transition functions. Since $\mathcal{O}$ has trivial transition functions, isomorphism classes of line bundles on S form a group called the ***Picard group of*** S.

We want to relate line bundles to meromorphic functions. For $p \in S$, there is a line bundle $L[p]$ whose sections is given by meromorphic functions which have at worst a simple pole at p and are holomorphic elsewhere. In terms of transition functions, we have a trivialising cover $S = D \cup (S \setminus \{p\})$ where D is a disc around p. Then our line bundle is determined by a holomorphic map $D \cap (S \setminus \{p\}) \to \mathbb{C}^*$, i.e. a nowhere vanishing function on $D \setminus \{p\}$. We can simply choose a local coordinate z for this, and check that changing the choice gives isomorphic bundles.

Definition 9.6. A divisor on a compact Riemann surface is a formal sum $D = \sum_{j=1}^n a_j p_j$ for $a_j \in \mathbb{Z}$, $p_j \in S$. Corresponding to this we have the line bundle $L[D] = \bigotimes_{j=1}^n L[p_j]^{\otimes a_j}$ where a negative power corresponds to taking duals first.

Divisors form an abelian group $\mathrm{Div}(S)$ with a group homomorphism $\deg : \mathrm{Div}(S) \to \mathbb{Z}$ sending $\sum_j a_j p_j \mapsto \sum_j a_j$. Moreover, the set of divisors has a partial order such that $D \geqslant E$ if all the coefficients of $D - E$ are non-negative.

If f is a meromorphic function with zeros of order a_j at p_j and poles of order b_j at q_j, we get a divisor $(f) = \sum a_j p_j - \sum b_j q_j$. Divisors arising in this way are called *principal*, and the principal divisors form a subgroup of $\mathrm{Div}(S)$. The quotient $\mathrm{Cl}(S)$ is called the *divisor class group*.

These structures satisfy the following additional properties.

1. Note that principal divisors have degree zero. Thus the degree map descends to give a group homomorphism $\mathrm{Cl}(S) \to \mathbb{Z}$.
2. Two divisors D_1, D_2 define isomorphic line bundles if and only if $D_1 - D_2$ is principal.
3. The global holomorphic sections of $L[D]$ correspond to meromorphic functions f satisfying $(f) \leqslant D$. Thus for $D = \sum a_j p_j$, the vector space $H^0(D)$ of meromorphic functions with at worst poles of order a_j at p_j is the space of global sections of $L[D]$. We also write $H^0(L[D])$ for $H^0(D)$.

The first of these implies that we have a map $\mathrm{Cl}(S) \to \mathrm{Pic}(S)$. This is in fact an isomorphism, see [Don04]. Thus all line bundles arise from divisors, and we can talk of the degree of a line bundle.

We can now restate Riemann–Roch as

$$\dim H^0(L) - \dim H^0(K - L) = \deg L - g + 1$$

where $K - L$ denotes $T^*S \otimes L^*$. To get this form of Riemann–Roch, we have used the identification of global sections with meromorphic functions and a similar result that sections of $K - L[D]$ are holomorphic 1-forms vanishing along D.

§ 9.2 Embeddings in projective space. Let $L \to S$ be a line bundle on a compact Riemann surface.

Definition 9.7. A *trivialisation* of L over an open set $U \subset S$ is a nowhere vanishing section of L on U.

Let $s_0, \ldots, s_n$ be a basis of global sections of L, and suppose for all $p \in S$ there is an s_j with $s_j(p) \neq 0$ ("L is basepoint-free"). Let ξ be a trivialisation of L over $U \subset S$, so $s_j = f_j \xi$ for holomorphic functions f_j on U. Thus we have a map $\varphi_L : U \to \mathbb{P}^n$, this is well-defined independent of trivialisation since another trivialisation would rescale all $f_j(p)$ by the same amount. In particular these glue across open sets, and we have a map $\varphi_L : S \to \mathbb{P}^n$ written $\varphi_L(p) = [s_0(p) : \ldots : s_n(p)]$.

Definition 9.8. We say L is very ample if φ_L is a closed immersion.

Theorem 9.9. *Every compact Riemann surface has a very ample line bundle. In particular, all compact Riemann surfaces are projective algebraic varieties.*

Sketch. The main idea is that once $\deg(D) \geqslant 2g$ then $L[D]$ is basepoint free, and once $\deg(D) \geqslant 2g + 1$ then $L[D]$ is very ample.

Take divisor D with degree $\geqslant 2g + 1$. Note $\deg K = 2g - 2$, since we showed that a holomorphic 1-form has $\chi(S) = 2g - 2$ zeros with multiplicity. Thus $\deg(K - D) \leqslant -1$ and there are no global holomorphic sections of $K - D$, giving us $H^0(K - L[D]) = 0$. Thus by Riemann–Roch we have $\dim H^0(L[D]) \geqslant g + 2$. Similarly for any point $p \in S$, we can show $\dim H^0(L[D - p]) < \dim H^0(L[D])$ i.e. not all sections of $L[D]$ vanish at p. Thus $L[D]$ is basepoint-free.

The proof of very-ampleness is analogous. It is then a very general fact (Chow's theorem) that all compact complex submanifolds of $\mathbb{P}^n$ admit the structure of projective varieties. For Riemann surfaces, the proof goes roughly as follows– if $S \subset \mathbb{P}^n$ is one dimensional then the homogeneous coordinates z_i induce meromorphic functions $f_j = \frac{z_j}{z_0}$ on S. These meromorphic functions realise S as a branched cover of $\mathbb{P}^1$, so that the field of meromorphic functions on S is a finite field extension of the corresponding field on $\mathbb{P}^1$. In particular any two meromorphic functions on S are related by a polynomial, so S is contained in the vanishing set of some polynomials. A little topology shows S is genuinely the vanishing locus of these polynomials. □

This marks the end of the course. With another lecture we would have looked at sheaf cohomology using Cech covers and explained Riemann–Roch in these terms, writing $\chi(L) := \dim H^0(L) - \dim H^1(L) = \deg L - g + 1$. This is proven through Serre duality, $H^0(K - L) \cong H^1(L)$. The motivation for cohomology is explained in [Don04, chapter 12], and this clarifies many constructions in the course– in particular how to view higher cohomology groups as obstructions.

References

[Don04] Simon Donaldson. "Riemann Surfaces". In: (Dec. 2004). url: `https://www.ma.imperial.ac.uk/~skdona/RSPREF.PDF` (cit. on pp. 17, 20, 23).