Introduction to hyperKähler Geometry

Based on a lecture series by Luca Giovenzana.

December 2023

HyperKähler and K3 manifolds, being interesting objects to study in their own right, also provide an excellent playground to study ideas in Hodge theory and deformation/moduli theory. In these introductory lectures we explore some of the classical theory surrounding the differential and algebraic geometry of these objects.

§ 0.1 About. This course on hyperKähler geometry was a part of a Winter School for students and early career researchers, organised by the UK Algebraic Geometry Network. The lectures were delivered in-person in the University of Warwick, and were transcribed by Parth Shimpi. These notes have undergone several amendments and are not a verbatim recall of the lectures, therefore discretion is advised when using this material. The notes are available online at `https://pas201.user.srcf.net/documents/2023-ukag-hyperkahler.pdf`. All errors and corrections should be communicated to by email to `parth.shimpi@glasgow.ac.uk`.

§ 1 Basics on K3 surfaces and hyperKähler manifolds

Definition 1.1. A compact complex surface X is a K3 if $H^1(X, \mathcal{O}_X) = 0$ and $\Omega^2_X \cong \mathcal{O}_X$.

Example 1.2. There are many examples of K3 surfaces, with explicit equations. For example, smooth quartic hypersurface $X \subset \mathbb{P}^3$ is K3. Indeed, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

so taking the long exact sequence of cohomology, we find $H^1(X, \mathcal{O}_X) = 0$. Secondly, by adjunction we see that $K_X = (K_{\mathbb{P}^3} + X)|_X = (-4H + 4H)|_X = 0$ so the canonical bundle of X is trivial. A similar argument shows that a smooth surface defined by a quadric and a cubic equation in $\mathbb{P}^4$ is K3. Likewise for three quadrics in $\mathbb{P}^5$.

Generalising the example of a quartic, if Y is Fano, then a smooth surface $X = D_1 \cap \ldots \cap D_n$ defined as an intersection of divisors is K3 if $\sum D_i + K_Y = 0$. Thus K3 surfaces often arise from fundamental linear systems of Fano 3-folds. As another example, if $\mathbb{V}(f_6) \subset \mathbb{P}^2$ is a smooth sextic, then $X = \mathbb{V}(y^2 - f_6) \subset \mathbb{P}(1,1,1,3)$ is a K3 surface in the anticanonical linear system of the weighted projective space.

Example 1.3. Another way to obtain K3 surfaces is as quotients of Abelian varieties. Consider a complex 2-dimensional torus A, and the map $(-1) : A \to A$ which sends an element to its inverse under the group law. The quotient $A/\{\pm 1\}$ is singular at precisely the image of the 2-torsion subgroup $A[2] \subset A$ which consists of sixteen points. Blowing up at these, we get a K3 surface called the *Kummer surface*.

The differential geometry of K3 surfaces is well understood– in particular they are all diffeomorphic as smooth $\mathbb{R}$-manifolds. Thus there is a unique lattice (i.e. a free abelian group equipped with a symmetric bilinear form) corresponding to the second integral cohomology of a K3 surface equipped with the intersection pairing, called the *K3 lattice*. Before stating its form, we recall some notation from lattice theory– we write $U \cong (\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ for the *hyperbolic lattice*. The E_8 lattice is characterised by its property of being the unique eight-dimensional lattice that is unimodular (i.e. has intersection matrix of determinant ± 1.) Another lattice that will appear later is I_1, the positive-definite one-dimensional unimodular lattice. The direct sum of two lattices is always orthogonal, and if $\Lambda = (\mathbb{Z}^{\oplus n}, M)$ is a lattice and m is an integer, then we define the *twisted lattice* $\Lambda(m)$ as $\mathbb{Z}^{\oplus n}$ equipped with the pairing mM.

Proposition 1.4. *If X is a K3 surface, then the singular cohomology $H^2(X, \mathbb{Z})$ equipped with the intersection pairing is isomorphic to the 22-dimensional lattice $\Lambda_{K3} = U^{\oplus 3} \oplus E_8^{\oplus 2}(-1)$. This has signature $(3, 19)$.*

The Torelli theorem for K3 surfaces asserts that the complex structure is determined by how this sits inside the complex cohomology. This will help study deformations and moduli of K3 surfaces later.

§ 1.1 HyperKähler manifolds. K3 surfaces form the two-dimensional counterpart of a larger family of manifolds with trivial canonical bundles and symplectic structures, called hyperKähler manifolds.

Definition 1.5. A complex manifold (M, I) is *Kähler* if there exists a metric g compatible with the complex structure I such that the $(1,1)$-form $\omega = g(I \cdot -, -)$ is closed. A compact Kähler manifold X is *hyperKähler* if it is simply connected (i.e. $\pi_1(X) = 0$) and the group $H^0(X, \Omega_X^2)$ is generated by a holomorphic symplectic form.

Example 1.6. $\mathbb{P}^n_{\mathbb{C}}$ is a Kähler manifold, with Kähler form given by the Fubini-Study metric $\omega = \frac{i}{2}\partial\bar{\partial}\log|z|^2$. Thus every projective manifold is also Kähler, and not much is lost by thinking of the Kähler condition as a mild generalisation of the projective condition.

Observe that the existence of a symplectic form $\sigma : T_X \otimes T_X \to \mathcal{O}_X$ on a hyperKähler manifold X implies the dimension of X is even. Moreover, can use σ to identify T_X with $\mathrm{Hom}(T_X, \mathcal{O}_X) = \Omega_X^1$. Thus $\sigma^{\wedge n} : \mathcal{O}_X \to \Omega_X^n$ gives a trivialisation of the canonical bundle.

Indeed, two dimensional hyperKähler manifolds are precisely K3 surfaces– by a result of Siu, all K3 surfaces (even the non-algebraic ones) are Kähler. Since all K3 surfaces are diffeomorphic, it suffices to show any one of them is simply connected and one does this for a smooth quadric in $\mathbb{P}^3$ using the hard Lefshetz theorem. Then the K3 condition implies any holomorphic global function gives a holomorphic symplectic form generating $H^0(X, \Omega_X^2)$.

In higher dimensions, there aren't many known examples for hyperKähler manifolds and the ones we know arise from K3s.

Example 1.7 (Hilbert schemes of K3 surfaces). Let S be a K3 surface and consider the product $S^n = S \times ... \times S$. The symmetric group Σ_n acts on this, and the quotient $S^{(n)} = S^n/\Sigma_n$ is a variety singular along the image of the *big diagonal* $\Delta = \{(x_1, ..., x_n) \in S^n \mid x_i = x_j \text{ for some } i \neq j\}$. Beauville proved that there exists a crepant resolution $S^{[n]} \to S^{(n)}$ (called the *Hilbert–Chow morphism*), and the variety $S^{[n]}$ (called the *Hilbert scheme*, also written $\mathrm{Hilb}^n(S)$) parametrises zero-dimensional subschemes of length n in S.

This is concretely realised in the special case $n = 2$, where we see that the Hilbert–Chow morphism $S^{[2]} \to S^2/\Sigma_2$ sends a length 2 subscheme to its support $Z \mapsto \sum_{P \in \mathrm{Supp} Z} \mathrm{length}\mathcal{O}_{Z,P} P$ and is the blow-up of the singular locus $\Delta \subset S^2/\Sigma_2$. The exceptional locus $E \cong \mathbb{P}_S(TS)$ is precisely the preimage of points of form $2P$.

More importantly for us, $S^{[n]}$ is hyperKähler. Indeed it is easy to show the existence of a holomorphic symplectic form– since S is K3, there is a non-trivial element $\sigma \in H^0(S, \Omega_S^2)$ and we will use this to construct one on the Hilbert scheme. If $\pi_i : S^n \to S$ denotes the ith projection, then $\pi_i^*\sigma$ is symplectic on S^n and $\sum_i \pi_i^*\sigma$ is invariant under the Σ_n action. Thus it descends to a symplectic form on $S^{(n)}$, and can be pulled back via the crepant resolution.

Example 1.8 (Generalised Kummer manifolds). If A is a 2-dimensional complex torus, there is a natural map $\Sigma : A^{(n)} \to A$ which applies the group law to a tuple $(a_1, ..., a_n) \in A^n$. Then considering the composite map $f : A^{[n]} \to A^{(n)} \to A$, the preimage $\mathrm{Kum}_{n-1}(A) := f^{-1}(0)$ is a $2(n-1)$ dimensional Kähler manifold called the *generalised Kummer manifold.*

More examples were provided by O'Grady, who constructed a 6-fold and a 10-fold as the resolutions of moduli spaces of sheaves on K3 surfaces and abelian surfaces.

§1.2 Other properties and names. HyperKähler surfaces are so called because they are very Kähler– we can use the Kähler structure (M, I, g) to find two more complex structures J, K such that g is a Kähler metric for all complex structures $aI + bJ + cK$ parametrised over the unit sphere $(a, b, c) \in S^2 \subset \mathbb{R}^3$.

HyperKähler manifolds are also called *irreducible holomorphically symplectic manifolds* (IHSMs), because of the result below.

Theorem 1.9 (Beauville–Bogomolov). *Let X be a compact Kähler manifold such that $c_1(X) = 0$. Then there exists a finite étale morphism $\tilde{X} \to X$ such that $\tilde{X} = T \times \prod_j Y_j \times \prod_j Z_j$, where the factor T is a complex torus, the Y_j are strict Calabi–Yau (i.e. $K_{Y_j} = 0$ and $H^0(Y_j, \Omega^p_{Y_j}) = 0$ for all $0 < p < \dim Y_j$) and the Z_j are hyperKähler.*

§2 Deformations of hyperKähler manifolds

Because the differential geometry of hyperKähler manifolds is so rigid, the only thing that can change in a family is the complex structure, which is largely controlled by the Hodge theory of these manifolds. Being an algebraic construction, this makes hyperKähler manifolds particularly suited for studying deformation and moduli problems.

§2.1 Recap: Hodge theory. Recall that a (pure) Hodge structure of weight k on a finitely generated Abelian group H is a decomposition $H \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$ such that $\overline{H^{p,q}} = H^{p,q}$ where $H \otimes \mathbb{C}$ has a canonical conjugation map given by $h \otimes \lambda \mapsto h \otimes \bar{\lambda}$.

If X is a complex manifold then we have $H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C}$, and the Hodge decomposition theorem states that $H^k(X, \mathbb{Z})$ has a pure Hodge structure of weight k whenever X is a compact Kähler manifold. The decomposition factors are given by $H^{p,q}(X) = H^q(\Omega_X^p)$, and the Hodge decomposition respects the ring structure on cohomology in that $H^{p,q}(X) \smile H^{r,s}(X) \subset H^{p+r,q+s}(X)$.

Note also that the subspace $H^{p,q}(X) \oplus H^{q,p}(X)$ is stable under complex conjugation, so considering $H^n(X, \mathbb{R})$ as a subspace of $H^n(X, \mathbb{C})$ we see that there is an induced decomposition $H^n(X, \mathbb{R}) = \bigoplus_{\substack{p+q=n \\ p<q}} H^{p,q}(X, \mathbb{R})$ where we write $H^{p,q}(X, \mathbb{R}) = H^n(X, \mathbb{R}) \cap (H^{p,q}(X) \oplus H^{q,p}(X))$.

§2.2 The cohomology of hyperKähler manifolds. We saw that the singular cohomology of a K3 surface is given by the K3 lattice, we will derive a similar structure if X is hyperKähler of dimension $2n$. In this situation, we have a symplectic form σ generating $H^0(X, \Omega_X^2)$, normalised so that $\int_X (\sigma\bar{\sigma})^n = 1$. This allows us to define a quadratic form on $H^2(X, \mathbb{Z})$ as

$$q_X(\alpha) = \frac{n}{2} \int_X \alpha^2 (\sigma\bar{\sigma})^{n-1} + (1-n) \left(\int_X \alpha \sigma^n \bar{\sigma}^{n-1} \right) \left(\int_X \alpha \sigma^{n-1} \bar{\sigma}^n \right).$$

Note if $n = 1$ (i.e. X is a K3 surface) then $q_X = \frac{1}{2} \int_X \alpha^2$ is proportional to the intersection form. This is true more generally, in the sense below.

Theorem 2.1. *If X is a hyperKähler manifold of dimension $2n$, then there exists a $C \in \mathbb{R}$ such that $\forall \alpha \in H^2(X, \mathbb{Z})$ we have $q_X(\alpha)^n = C \int_X \alpha^{2n}$. In particular, q_X can be rescaled to obtain an integral and primitive non-degenerate form $H^2(X, \mathbb{Z}) \to \mathbb{Z}$ called the* Beauville–Fujiki form *which has signature $(3, b_2(X) - 3)$.*

We will sketch explicit subspaces on which the form is positive and negative definite. Since $H^{2,0}(X)$ is spanned by a holomorphic symplectic form σ, any $\alpha \in H^2(X, \mathbb{C})$ is of the form $\alpha = \lambda\sigma + \beta + \mu\bar{\sigma}$ for $\beta \in H^{1,1}(X)$ and we then have $q_X(\alpha) = \lambda\mu + \frac{n}{2} \int_X \beta^2 (\sigma\bar{\sigma})^{n-1}$. In particular $H^{1,1}(X)$ is orthogonal to $H^{2,0}(X) \oplus H^{0,2}(X)$ and q_X is positive definite on all forms $\alpha = \lambda\sigma + \overline{\lambda\sigma}$. The formula also shows that q_X is positive definite on any Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R})$, so q_X is positive definite on the three dimensional subspace $H^{0,2}(X, \mathbb{R}) \oplus \mathbb{R}[\omega] \subset H^2(X, \mathbb{R})$.

To understand the orthogonal subspace on which the form is negative definite, recall that (the cohomology class of) a k-form α is *primitive* if $\omega^{n-k+1} \cdot \alpha = 0$. It follows that a 2-form α is primive if and only if $q_X(\alpha, \omega) = 0$, and hence q_X is negative definite precisely on the space $H^{1,1}(X, \mathbb{R})_{\text{prim}}$ of primitive real (1,1)-forms.

Example 2.2. If $X = S^{[n]}$ for S a K3 surface, then $H^2(X, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus I_1(2-2n)$ where the lattice $I_1(2-2n)$ is generated by $\frac{1}{2}[E]$ for E the exceptional divisor of the Hilbert–Chow morphism $S^{[n]} \to S^{(n)}$. On the other hand if $X = \mathrm{Kum}_n(A)$ for a 2-dimensional complex torus A, we have $H^2(X, \mathbb{Z}) = U^{\oplus 3} \oplus I_1(-2-2n)$.

Remark 2.3. For every known example, the form $q_X : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ is even (i.e. has image in $2\mathbb{Z}$) but the proof for general hyperKähler manifolds remains open.

§2.3 Deformation families. We can now discuss the main content of this section, the study of hyperKähler manifolds in families.

Definition 2.4. A *deformation* of a manifold X is a smooth proper morphism of complex spaces $\mathcal{X} \to S$, where $s \in S$ is a distinguished point and $\mathcal{X}|_s \cong X$. The deformation is said to be *infinitesimal* if $S = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$.

A deformation $\mathcal{X} \to S$ is said to be *universal* if for any other deformation $\mathcal{X}' \to S' \ni s'$, there is a unique map $(S', s') \to (S, s)$ such that $\mathcal{X}' \cong \mathcal{X} \times_S S'$.

Remark 2.5. Note we are working with smooth (not just flat) deformations, and this allows a lot of differential geometric tools to come into play. In particular by Ehresmann's theorem, whenever $\mathcal{X} \to S$ is a deformation of X, we have an open cover $S = \bigcup_\alpha U_\alpha$ such that $\mathcal{X}|_{U_\alpha}$ is diffeomorphic to $X \times U_\alpha$. Thus only the complex structure is deformed.

Theorem 2.6. *The set of infinitesimal deformations of X up to equivalence is in bijection with the space $H^1(X, T_X)$.*

Sketch. Let $\mathcal{X} \to \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ be an infinitesimal deformation. By Ehressmann's theorem the normal bundle of the central fiber X is trivial, hence the normal sequence

$$0 \to T_X \to T_{\mathcal{X}}|_X \to \mathcal{O}_X \to 0$$

gives an element in $\mathrm{Ext}^1(\mathcal{O}_X, T_X) = H^1(X, T_X)$. The inverse map sends an element $\nu \in H^1(X, T_X)$ represented by a 1-cocycle $\{\nu_{ij} \in H^0(U_i \cap U_j, T_X)\}$ to the family $\mathcal{X} \to \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ constructed by gluing the trivial deformations $U_i \times \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ along the maps $(\mathrm{id} + \epsilon\nu_{ij}) : \mathcal{O}_{U_i \cap U_j}[\epsilon] \to \mathcal{O}_{U_i \cap U_j}[\epsilon]$. □

Thus the set of infinitesimal deformations is naturally a vector space. It is in fact the tangent space to the base of the universal deformation $\mathcal{X} \to \mathcal{D}ef X \ni 0$, if it exists. Indeed in this case infinitesimal deformations are given by pointed morphisms $\operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2) \to \mathcal{D}ef X$, and the space of such maps $\mathrm{Mor}_*(\operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2), \mathcal{D}ef X)$ is the the tangent space $T_0 \mathcal{D}ef X$.

We will now see that hyperKähler manifolds always have universal deformations and locally they are simply $\mathbb{C}^n$.

Theorem 2.7. *For any complex manifold X, if $H^0(X, T_X) = 0$ then a universal deformation $\mathcal{X} \to \mathcal{D}ef X$ exists. Moreover for all $t \in \mathcal{D}ef X$, this is a universal deformation of $X_t = \mathcal{X}|_t$.*

Corollary 2.8. *HyperKähler manifolds have universal deformations.*

Proof. If X is a hyperKähler manifold, then the symplectic form $\sigma : T_X \times T_X \to \mathcal{O}_X$ induces an isomorphism $T_X \cong \Omega_X$ and hence $H^0(X, T_X) = H^0(X, \Omega_X)$. By Hodge symmetry this has the same dimension as $H^1(X, \mathcal{O}_X)$, but this space vanishes for hyperKähler manifolds. □

Theorem 2.9 (Bogomolov–Tian–Todorov). *If X is hyperKähler, then the universal deformation space $\mathcal{D}ef X$ is smooth (and hence equal to $H^1(X, T_X)$).*

Example 2.10. If S is a K3 surface, then note that $h^1(S, T_S) = h^{1,1}(S) = 20$. Can all deformations arise as smooth quartics? Note a quartic equation has $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) = 35$ parameters. On the other hand the automorphism group $\mathrm{PGL}(4, \mathbb{C})$ is 15 dimensional, so the space of quartics in $\mathbb{P}^3$ has dimension $35-1-15 = 19$. Thus we conclude that S must admit deformation which does not embed in $\mathbb{P}^3$.

§ 3 Moduli of hyperKähler manifolds

We have seen that if X is a hyperKähler manifold then it has a universal deformation over $\mathcal{D}ef X = H^1(X, T_X)$ centered at the origin. Moreover $\mathcal{D}ef X$ has an open subset U_X such that the family restricted to U_X is a universal deformation of every fiber $X|_t$ ($t \in U_X$). Thus we can naïvely consider the set

$$\mathcal{M}_X = \{Y \mid Y \text{ is hyperKähler and deformation equivalent to } X\}$$

and use the charts from each $U_Y \subset \mathcal{D}ef Y \cong \mathbb{C}^{b_2-2}$ to give this a complex structure, since if $Z \in U_Y$ then there is an open subset of U_Z identified with a neighbourhood of Z in U_Y by the universality. The same argument also guarantees compatibility at $Z \in U_Y \cap U_{Y'}$, since the corresponding restricted universal families $\mathcal{Y}|_{U_Y \cap U_{Y'}}$ and $\mathcal{Y}'|_{U_Y \cap U_{Y'}}$ are both also universal deformations of Z. Thus the set of all hyperKähler manifolds $\mathcal{M}_X$ has the structure of a complex space, but it turns out this is not Hausdorff and hence isn't a manifold.

To get a well behaved moduli space, we hence parametrise hyperKähler manifolds with extra data, namely the choice of an isomorphism $H^2(X, \mathbb{Z}) \cong \Lambda_{K3}$ and (later) the choice of an ample bundle. For the rest of the section we will build these ideas for the specific case of a K3 surface, but everything generalises to hyperKähler manifolds but with more bookkeeping.

§ 3.1 Moduli of marked K3 surfaces.

As mentioned before, K3 surfaces have a Torelli theorem which reduces the problem to algebraic data.

Theorem 3.1 (Torelli theorem for K3 surfaces). *Two K3 surfaces X, X' are isomorphic if and only if there is a Hodge isometry $H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$. Moreover, such an isometry γ is of the form $\gamma = f^*$ for $f : X \to X'$ if and only if there is a Kähler class $\omega_X \in H^2(X, \mathbb{R})$ such that $\gamma(\omega_x) \in H^2(X', \mathbb{R})$ is Kähler.*

Definition 3.2. A *marked K3 surface* is a pair (X, φ), where X is a K3 surface and $\varphi : H^2(X, \mathbb{Z}) \to \Lambda_{K3}$ is an isometry. An isomorphism $(X, \varphi) \to (X', \varphi')$ is an isomorphism $f : X \to X'$ such that $\varphi \circ f^* = \varphi'$.

By the Torelli theorem, parametrising marked K3 surfaces is equivalent to parametrising Hodge structures on Λ_{K3}. Such a structure is given by a filtration $0 \subset F^0 \subset F^1 \subset \Lambda_{K3} \otimes \mathbb{C}$ where $\dim F^0 = 1$, $\dim F^1 = 21$. Thus to first approximation the parameter space is in the variety $\mathrm{Flag}(1, 21, \Lambda_{K3})$. But this space has dimension $21 \times 20 = 420$, which is much larger than the expected dimension of $\mathcal{D}ef X$!

Indeed not all filtrations can arise– the Hodge structure in this situation is completely determined by $H^{2,0}(X)$, since $H^{0,2}(X) = \overline{H^{2,0}}(X)$ and $H^{1,1} = (H^{2,0}(X) \oplus H^{0,2}(X))^{\perp}$. Moreover if σ is a generator of $H^0(X, \Omega^2_X)$, then it satisfies the *Hodge–Riemann bilinear relations* $(\sigma, \bar{\sigma}) > 0$, $(\sigma, \sigma) = 0$. Thus our desired parameter space is a subset of the Grassmannian $\mathrm{Gr}(1, 22) = \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$ given by

$$\Omega_{K3} = \{[x] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid (x, x) = 0 \text{ and } (x, \bar{x}) > 0\},$$

called the *period domain.* This is a 20-dimensional complex manifold, given by a Euclidean open set in the subvariety defined by a quadratic form in $\mathbb{P}^{21}$.

Write $\mathcal{M}_{K3}$ for the set of marked K3 surfaces up to isomorphism. By construction, sending a marked K3 surface (X, φ) to the point $[\varphi(H^{2,0}X)] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$ gives a (surjective!) map $\mathcal{M}_{K3} \to \Omega_{K3}$, called the *period map.* In fact we can give $\mathcal{M}_{K3}$ an analytic structure by considering families of marked K3 surfaces.

Definition 3.3. A *family of marked K3 surfaces* is given by a smooth map $\pi : \mathcal{X} \to S$ and an isometry of sheaves $\Phi : \mathbf{R}^2\pi_*\underline{\mathbb{Z}} \xrightarrow{\sim} \underline{\Lambda_{K3}}$.

The *period map associated to a family* $(\mathcal{X} \to S, \Phi)$ is given by the map $\mathcal{P} : S \to \Omega_{\Lambda_{K3}}$, which sends $s \in S$ to the "period of $\mathcal{X}|_s$", i.e. the subspace $[\Phi_s(H^{2,0}(\mathcal{X}|s))]$.

Thus the period map for families gives charts on $\mathcal{M}_{K3}$, and the global period map $\mathcal{M}_{K3} \to \Omega_{K3}$ is holomorphic. It not injective– if (X, φ) and (X', φ') have the same periods, then theree is an isomorphism $f : X \to X'$ by the Torelli theorem but φ may differ from $f^* \circ \varphi'$ by an automorphism of the lattice (in particular it is possible that such an automorphism that does not preserve the Kähler cone and hence the isometry isn't realised as a pullback). Thus it makes sense to quotient by the action of the orthogonal group $O(\Lambda_{K3})$ to obtain a map

$$\mathcal{M}_{K3}/O(\Lambda_{K3}) \longrightarrow \Omega_{K3}/O(\Lambda_{K3}).$$

By the Torelli theorem this map is an isomorphism of complex spaces, and the points of this orbit space are isomorphism classes of K3 surfaces.

The connected components of $\mathcal{M}_{K3}/O(\Lambda_{K3})$ are the moduli spaces of K3 surfaces that belong to a particular deformation family, this is precisely the construction that came from considering universal deformations at the start of this section. In this context, the non–Hausdorff behaviour can be seen as coming from the fact that the action of $O(\Lambda_{K3})$ is far from being properly discontinuous.

§ 3.2 Algebraicity of K3 surfaces. The moduli space constructed above is ill-behaved partly because we are considering 'too many' surfaces– most K3 surfaces in the moduli space are not algebraic.

Theorem 3.4 (Kodaira). *A complex manifold X is projective if and only if $H^2(X, \mathbb{Z})$ contains a Kähler class.*

Indeed by the Lefshetz $(1, 1)$-theorem we know that every integral $(1, 1)$-class (i.e. element of $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$) is the Chern class of some holomorphic line bundle, and Kodaira's theorem shows that the classes with positive representatives (i.e. Kähler classes) come from ample line bundles.

Corollary 3.5. *A very general*[1] *K3 surface is not projective.*

Proof. If X is a projective K3 surface, then $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is necessarily non-zero and hence the subspace $H^{2,0}(X)$ (which determines the complex structure on X) is orthogonal to some $\delta \in H^2(X, \mathbb{Z})$. It follows that in Ω_{K3}, the locus of marked projective K3 surfaces is contained in the closed hyperplane arrangement $\Omega_{K3} \cap \left(\bigcup_{\delta \in \Lambda_{K3}} \delta^\perp\right)$. The complement U is dense and open, and any K3 surface with period in U is not projective. ☐

Thus to get better behaved moduli spaces, we consider polarised K3 surfaces.

Definition 3.6. A *polarised K3 surface* is a pair (X, L) where X is a K3 surface and L is a primitive (i.e. not of the form $M^{\otimes(>1)}$) ample line bundle. The *degree* of the polarised K3 surface (X, L) is equal to $\deg L = L^2$.

§ 3.3 Moduli of polarised K3 surfaces. Note the degree of a polarised K3 surface X is always even and positive, say equal to 2d. Now in this situation $L^{\otimes 3}$ is very ample, giving an embedding $X \hookrightarrow \mathbb{P}^{9d+1}$ such that X has Hilbert polynomial $f(t) = 9dt^2 + 2$. Thus the straightforward thing to do is to realise that polarised K3 surfaces of degree 2d lie in an open subset of the Hilbert scheme $\mathrm{Hilb}^{9dt^2+2}(\mathbb{P}^{9d+1})$, and thus the required moduli space of K3 surfaces is obtained as a GIT quotient of this locus by the $\mathrm{PGL}(9d + 1, \mathbb{C})$ action. In practice, however, checking for semistability is hard.

This is circumvented by constructing the moduli space explicitly from that of marked K3 surfaces and using a stronger form of the Torelli theorem. We will provide a sketch.

[1] We use 'very general' to mean 'outside the union of countably many Zariski-closed subvarieties'.

By standard lattice theory (Eichler's criterion), up to the action of $O(\Lambda_{K3})$ there is a unique primitive element $\ell \in \Lambda_{K3}$ satisfying $\ell^2 = 2d$ and hence we can choose a marking $\varphi : H^2(X, \mathbb{Z}) \to \Lambda_{K3}$ with $\varphi(c_1 L) = \ell$. Then its reasonable to consider the set

$$\mathcal{M}_{2d} = \{(X, \varphi, L) \mid (X, \varphi) \text{ is a marked K3 surface, } (X, L) \text{ is polarised, and } \varphi[L] = \ell\}.$$

Now if $(X, \varphi, L) \in \mathcal{M}_{2d}$, then clearly the period point $[\varphi(H^{2,0}X)] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$ lies in the hyperplane $\{[x] \mid x \perp \ell\}$, which we can write as the projective space $\mathbb{P}(\Lambda_{2d} \otimes \mathbb{C})$ for $\Lambda_{2d} = \ell^\perp \cong U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus I_1(-2d)$. Thus the natural period domain to consider is

$$\Omega_{2d} = \{[x] \in \mathbb{P}(\Lambda_{2d} \otimes \mathbb{C}) \mid (x, x) = 0,\ (x, \bar{x}) > 0\}.$$

Turns out this has two irreducible components which are interchanged by complex conjugation, so we take the orientation preserving component Ω^+_{2d}. This is an example of a *Seigel upper half plane*, a family of spaces forming higher dimensional analogue of the upper half plane $\mathbb{H}$. There is a natural period map $\mathcal{M}_{2d} \to \Omega_{2d}$, and considering families of marked polarised K3 surfaces (defined in the obvious way) give $\mathcal{M}_{2d}$ a holomorphic structure.

Again, to forget the marking we consider the action of $\mathrm{Stab}(\ell) \subset O(\Lambda_{K3})$ on $\mathcal{M}_{2d}$ and that of $\mathrm{Stab}(\ell)^+$ on Ω^+_{2d} (where we take the subgroup preserving the chosen component). It is convenient to write $\Omega_{2d}/\mathrm{Stab}(\ell)$ for $\Omega^+_{2d}/\mathrm{Stab}(\ell)^+$, this is an irreducible normal quasi-projective variety by a theorem of Bailey–Borel. Then one way to state the *strong Torelli theorem for K3 surfaces* is that the induced period map $\mathcal{M}_{2d}/\mathrm{Stab}(\ell) \to \Omega_{2d}/\mathrm{Stab}(\ell)$ is an open immersion. This gives us the moduli of polarised K3 surfaces as a 19 dimensional quasi-projective variety.

Note the difference from the moduli space of marked K3 surfaces we constructed previously– the period map here not surjective. To compute the complement, recall that the Nakai–Kleiman criterion states a line bundle L on a surface X is ample if and only if $L^2 > 0$ and $L \cdot C > 0$ for all curves $C \subset X$. When X is K3, it suffices to test the second condition only on curves with self-intersection -2, i.e. the *roots* of Λ_{2d}. Writing $\Delta \subset \Lambda_{2d}$ for the set of roots, we thus see that the complement $(\Omega_{2d}/\mathrm{Stab}(\ell)) \setminus (\mathcal{M}_{2d}/\mathrm{Stab}(\ell))$ is given by the root hyperplanes $\bigcup_{\delta \in \Delta}\{[x] \mid x \perp \delta\}$.

Remark 3.7. For general hyperKählers there is a very similar picture that realises all deformation families. The slight subtlety is that the polarisation $\ell \in \Lambda_X$ is no longer uniquely determined, and hence it is also not obvious what the replacement for the group action should be. Turns out the solution is to consider the group of Hodge isometries arising as monodromies, but that is beyond the scope of these notes.

§4 Cubic 4-folds

In this section we will use some of the techniques discussed to address rationality problems for cubic 4-folds. To begin, note that the locus of smooth cubic 4-folds forms an open set V in the linear system $|\mathcal{O}_{\mathbb{P}^5}(3)| \cong \mathbb{P}^{55}$. Since the automorphism group $PGL(6, \mathbb{C})$ is 35-dimensional, we see that the quotient $\mathcal{C} = V /\!\!/ PGL(6, \mathbb{C})$ is a 20-dimensional quasi-projective variety, the *moduli space of cubic 4-folds*. Conjecturally rational cubics form a codimension one locus related intricately with the Hodge theoretic and derived=categorical behaviour of K3 surfaces.

Before proceeding, we first recall some facts about the cohomology of cubic 4-folds. If $Y \subset \mathbb{P}^5$ is a cubic hypersuface, its Hodge diamond looks like

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0. \end{array}$$

The most interesting bit is the middle cohomology lattice $H^4(Y, \mathbb{Z}) \cong I_1^{\oplus 21} \oplus I_1(-1)^{\oplus 2}$. We can pick a generator $h \in H^2(X, \mathbb{Z})$ associated (via the Chern class map) to a hyperplane section, then the class $h^2 \in H^4(X, \mathbb{Z})$ has self intersection 3. This allows us to define the *primitive cohomology lattice* $H^4(Y, \mathbb{Z})_{\mathrm{prim}} = (h^2)^\perp \cong E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

Note the similarity with the K3 lattice– in fact the primitive cohomology of certain cubic 4-folds can be identified with a subspace of the cohomology of an associated K3 surface, so that these cubics 'look like K3 surfaces' to Hodge theory. Conjecturally these are precisely the rational cubics. We will sketch this with examples.

§ 4.1 Explicit constructions. We look at two examples of rational cubic 4-folds and how K3 surfaces naturally arise in this setting.

Example 4.1 (Cubics containing two planes). Consider the cubic polynomials $g = u^2x + v^2y + w^2z$ and $h = ux^2 + vy^2 + wz^2$, and the associated hypersurface $Y = \mathbb{V}(g - h) \subset \mathbb{P}^5$. This is smooth, and one can consider the contained planes $\Pi_1 = \mathbb{V}(u, w, v)$ and $\Pi_2 = \mathbb{V}(x, y, z)$. Given two general points $p_1 \in \Pi_1$, $p_2 \in \Pi_2$, the line through them

$$\ell_{p_1,p_2} = \{s \cdot p_1 + t \cdot p_2 \mid [s : t] \in \mathbb{P}^1\}$$

intersects Y in three points $\{p_1, p_2, q\}$ by Bezout's theorem; this defines a rational map $\Pi_1 \times \Pi_2 \dashrightarrow Y$ given by $(p_1, p_2) \mapsto q$. It is a birational map– given any point $q \in \mathbb{P}^5 \setminus (\Pi_1 \cup \Pi_2)$, there is a unique line passing through q that meets Π_1 and Π_2. Thus Y is birational to $\mathbb{P}^2 \times \mathbb{P}^2$, i.e. it is rational.

The given map $\Pi_1 \times \Pi_2 \dashrightarrow Y$ is undefined whenever ℓ_{p_1,p_2} lies inside Y, i.e. on $S = \mathbb{V}(g, h) \cap \Pi_1 \cap \Pi_2$. Since S is a $(1, 2), (2, 1)$ complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$, it is a K3 surface.

Example 4.2 (Pfaffian cubics). Fix a vector space $V \cong \mathbb{C}^6$ and consider the Grassmannian $G = \mathrm{Gr}(2, V)$ parametrising lines in $\mathbb{P}(V)$. This is 8-dimensional, sits as a degree 14 subvariety inside $\mathbb{P}(\bigwedge^2 V)$ via the Plücker embedding, and is Fano with canonical bundle $\mathcal{O}_G(-6)$. Thus for a general 9-dimensional linear subspace $L \subset \bigwedge^2 V$, the variety $S = G \cap \mathbb{P}(L)$ is the complete intersection of G with six hyperplanes (corresponding to a basis of $L^\perp \subset \bigwedge^2 V^*$), and adjunction formula shows S is a K3 surface. In fact all general polarised K3 surfaces (S, H) of degree $H^2 = 14$ arise in this fashion [see Muk88, §3].

There is a natural way to produce cubic 4-folds in this setting. Note that the projective space $\mathbb{P}(\bigwedge^2 V^*)$ parametrises alternating forms $V \times V \to \mathbb{C}$ up to scaling. Now the determinant of a 6×6 skew-symmetric matrix M has form $\det M = \mathrm{Pf}(M)^2$ for a homogeneous cubic polynomial $\mathrm{Pf} \in \mathbb{C}[M_{ij} \mid 1 \leqslant i < j \leqslant 6]$, so the vanishing locus $\mathbb{V}(\mathrm{Pf}) \subset \mathbb{P}(\bigwedge^2 V^*)$ parametrises degenerate forms (i.e. forms of rank $\leqslant 4$). On the other hand the Grassmannian $G^* = \mathrm{Gr}(2, V^*)$ parametrises forms of rank 2, thus giving us a natural stratification $G^* \subset \mathbb{V}(\mathrm{Pf}) \subset \mathbb{P}(\bigwedge^2 V^*)$. If the choice of L was sufficiently general, then $L^\perp \subset \bigwedge^2 V^*$ is disjoint from G^* and is not contained in $\mathbb{V}(\mathrm{Pf})$. Thus $\mathbb{P}(L^\perp)$ is a $\mathbb{P}^5$ whose points correspond to skew-forms $V \times V \to \mathbb{C}$ of rank $\geqslant 4$. The locus of degenerate forms in $\mathbb{P}(L^\perp)$ is then given by a cubic hypersurface $Y = \mathbb{P}(L^\perp) \cap \mathbb{V}(\mathrm{Pf})$, called the *Pfaffian cubic* associated to L. If explicit coordinates are fixed, then the choice of $L^\perp$ corresponds to the choice of a 6×6 skew-symmetric matrix with entries given by linear forms in $\mathbb{C}[u, v, w, x, y, z]$, and Y cuts out the locus where this matrix is degenerate.

Exercise 4.3. Show that Y is rational, by considering a general 5-dimensional subspace $W \subset V$ and mapping $[\varphi] \in Y$ to $[W \cap \ker \varphi] \in \mathbb{P}(W)$. The birational map $\mathbb{P}^4 \dashrightarrow S$ is in fact a blow-up in a surface birational to S [see Has00, §4.1.3].

Thus to the data of $L \subset \bigwedge^2 V$, we have associated a cubic 4-fold Y parametrising non-degenerate alternating forms in $L^\perp$ and a K3 surface S parametrising the 2-planes $P \subset V$ such that $\varphi|_P = 0$ for every $\varphi \in L^\perp$. There is a surprising connection between the geometry of the two, first explored by Beauville and Donagi in [BD85]. They were studying the so-called *Fano variety of lines* $F(Y)$, which parametrises lines contained in Y. Since Y lies in $\mathbb{P}(L^\perp)$, the variety $F(Y)$ naturally sits in the Grassmannian $\mathrm{Gr}(2, L^\perp)$.

Theorem 4.4 (Beauville–Donagi). *The Fano variety of lines* $F(Y)$ *is isomorphic to the Hilbert scheme* $S^{[2]}$.

Sketch. Given two distinct points in S, we can realise them as 2-planes $P, Q \subset V$. By the generality hypothesis $P + Q$ is a 4-plane, and we consider the linear subspace $\ell = \{[\varphi] \in \mathbb{P}(L^\perp) \ : \ \varphi|_{P+Q} = 0\}$. This is 1-dimensional, and if $[\varphi] \in \ell$ then $\varphi|_P = \varphi|_Q = 0$ so φ is necessarily degenerate. Thus we have a line $\ell \subset Y$, and our map $S^{[2]} \to F(Y)$ maps $\{P, Q\} \mapsto \ell$. □

Remark 4.5. Despite its name, $F(Y)$ is not a Fano variety. Indeed the theorem shows it is hyperKähler, and the name is simply a reflection of the fact that this variety was first extensively studied by Fano.

§ 4.2 Hodge theory of cubic 4-folds. If $Y \subset \mathbb{P}^5$ is a cubic 4-fold, we can analogously construct the Fano variety of lines $F(Y) \subset \mathrm{Gr}(2, 6)$, this is a smooth hyperKähler 4-fold. There is a universal line $\mathcal{L} \subset F(Y) \times Y$ parametrising pairs (ℓ, y) such that $y \in \ell$. The natural map $q : \mathcal{L} \to F(Y)$ is a $\mathbb{P}^1$–bundle, and the fibers of the map $p : \mathcal{L} \to Y$ are lines in $\mathrm{Gr}(2, 6)$. There is an induced morphism on cohomology

$$\alpha = q_* \circ p^* : H^4(Y, \mathbb{Z}) \to H^2(F_1(Y), \mathbb{Z})$$

called the *Abel-Jacobi map*, which sends the class h^2 to $g = [\mathcal{O}_{F(Y)}(1)]$ where the hyperplane class on $F(Y)$ is induced from the Plücker embedding of the ambient Grassmannian. This map in fact preserves the Hodge structures, up to a twist.

Theorem 4.6. *The map* $\alpha : H^4(Y, \mathbb{Z}) \to H^2(F(Y), \mathbb{Z})(-1)$ *is an isomorphism of Hodge structures, and induces a Hodge isometry* $(h^2)^\perp \cong g^\perp$.

Now [Cha12] shows that $F(Y)$ in fact determines Y. Since hyperKähler manifolds have a Torelli theorem, we can use this construction to obtain an analogous one for cubic 4-folds.

Corollary 4.7. *Two cubic 4-folds are isomorphic if and only if their primitive cohomology lattices are isometric as Hodge structures.*

Thus fixing the lattice $M = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, we can consider the manifold Ω_M parametrising compatible Hodge structures on M. As before, there is a period map $\mathscr{P} : \mathcal{C} \to \Omega_M/O(M)$ from the moduli space $\mathcal{C}$ of cubic 4-folds. The Torelli theorem says this period map is an open immersion.

§ 4.3 Rationality problems and special cubics. In particular, the period map can be used to show that a very general cubic 4-fold has $H^{2,2}(X, \mathbb{Z}) \cong \mathbb{Z} \cdot h^2$.

Definition 4.8. Say a cubic 4-fold Y is *special* if there is a surface $S \subset Y$ with $[S] \notin \mathbb{Z} \cdot h^2 \subseteq H^{2,2}(X, \mathbb{Z})$.

For such Y, we say a *labelling* is a saturated rank 2 sublattice $K \subset H^{2,2}(X, \mathbb{Z})$ containing $\mathbb{Z} \cdot h^2$. The *discriminant* of the labelling is the determinant of the Gram matrix of the lattice.

Say a polarised K3 surface (X, L) is *associated to* a labelled cubic 4-fold (Y, K) if $K^\perp \subset H^4(Y, \mathbb{Z})$ is isometric as a Hodge structure to $[L]^\perp \subset H^2(X, \mathbb{Z})$.

These were extensively studied in [Has00], and we summarise some of the results.

Theorem 4.9 (Hasset). *For* $d \geqslant 0$*, consider the space* $\mathcal{C}_d \subset \mathcal{C}$ *of smooth cubic 4-folds that admit a labelling of discriminant* d*. Then* $\mathcal{C}_d$ *is an irreducible divisor if and only if* $d > 6$ *and* $d \equiv 0, 2 \pmod 3$

Theorem 4.10 (Hasset). *A labelled cubic 4-fold* (Y, K) *has an associated K3 surface if and only if its discriminant* d *is positive,* $4, 9 \nmid d$*, and all odd prime factors of* d *are of the form* $3n + 1$.

Conjecturally, rationality is equivalent to having an associated K3 surface.

Conjecture. (Hasset) A cubic 4-fold Y is rational if and only if $Y \in \mathcal{C}_d$ for d satisfying the conditions of the theorem above. In particular, a very general cubic 4-fold is irrational.

On the other hand, by considering *non-commutative K3 surfaces* arising as Kuznetsov components in $\mathbf{D}^b Y$, Kuznetsov conjectures another criterion for rationality.

Conjecture. (Kuznetsov) A cubic 4-fold Y is rational if and only if the Kuznetsov component associated to the exceptional collection $(\mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2))$ is equivalent to the derived category of a K3 surface.

References

[BD85] Arnold Beauville and Ron Donagi. "La Variété Des Droites d'une Hypersurface Cubique de Dimension 4". In: *C.R. Acad. Sc. Paris.* 1st ser. 301 (1985), pp. 703–706. url: https://math.univ-cotedazur.fr/~beauvill/pubs/bd.pdf (cit. on p. 7).

[Cha12] François Charles. *A Remark on the Torelli Theorem for Cubic Fourfolds.* Sept. 2012. arXiv: 1209.4509 [math]. url: http://arxiv.org/abs/1209.4509 (visited on 01/03/2024) (cit. on p. 8).

[Has00] Brendan Hassett. "Special Cubic Fourfolds". In: *Compositio Mathematica* 120.1 (2000), pp. 1–23. issn: 0010437X. doi: 10.1023/A:1001706324425. url: http://www.journals.cambridge.org/abstract_S0010437X00000038 (visited on 01/03/2024) (cit. on pp. 7, 8).

[Muk88] Shigeru Mukai. "Curves, K3 Surfaces and Fano 3-Folds of Genus ⩽ 10". In: *Algebraic Geometry and Commutative Algebra.* Elsevier, 1988, pp. 357–377. isbn: 978-0-12-348031-6. doi: 10.1016/B978-0-12-348031-6.50026-7. url: https://linkinghub.elsevier.com/retrieve/pii/B9780123480316500267 (visited on 01/03/2024) (cit. on p. 7).