§ Appendix: Algebraic groups and Lie theory (over $k=\bar{k}$ of characteristic 0)

- An algebraic group is an affine group-scheme $/k$, automatically smooth (Cartier's theorem). Such a group G acts on its coordinate ring $k[G]$ (do not confuse with the group ring kG), this action is faithful so it is faithful on some finite dimensional subrepresentation $V\subset k[G]$; for this use the fact that every representation is a colimit of its finite dimensional subrepresentations. Thus G is a (closed) subgroup of $GL(V)$, conversely every subgroup of GL_n is clearly an algebraic group.

- Standard notions all hold true. A map $H\to G$ of algebraic groups defines H as a subgroup of G if it defines an inclusion of schemes, it is automatically a closed immersion. A map $G\to H$ of algebraic groups defines H as a quotient of G if, as a morphism of schemes, it is dominant/surjective/faithfully flat (all conditions equivalent). The quotient $G\twoheadrightarrow H$ is split if there is a subgroup $H\to G$ such that the composite $H\to G\twoheadrightarrow H$ is an isomorphism; in this case G is the semidirect product $G=N\rtimes H$ where $N=\ker(G\to H)$.

- The group G is solvable if it can be built from commutative groups by extensions, ie G has subgroups $G_0,\dots,G_n$ satisfying $G=G_0\supset\dots\supset G_n=1$ with G_i normal in G_{i+1}, and G_{i+1}/G_i commutative. If G is connected, solvability is equivalent to there being a subnormal series like above but with each G_i normal in G, and $G_i/G_{i+1}\cong\mathbb{G}_m$ or $\mathbb{G}_a$.

  ... nilpotent if it can be built from commutative groups by central extensions, ie the subnormal series above additionally satisfies $G_i\lhd G$, and $G_i/G_{i+1}\subset\mathrm{Center}(G/G_{i+1})$. If G is connected, it is nilpotent iff it is solvable and some (equivalently all) maximal torus is central.

  ... unipotent if it can be built from copies of $\mathbb{G}_a$ by central extensions, ie all G_i/G_{i+1} in the central series above are isomorphic to $\mathbb{G}_a$. Equivalently, G contains no copy of $\mathbb{G}_m$ (equivalently $\mathbb{1}$ is a maximal torus in G). Equivalently every non-zero (or some faithful) representation V has a non-zero trivial subrepresentation. Equivalently for every $G\xrightarrow{\varphi}GL_n$, each φg ($g\in G$) is a unipotent matrix (ie $(\varphi g-1)^{\gg 0}=0$). Equivalently every $G\xrightarrow{\varphi}GL_n$ has image, up to conjugation, in the group $\mathbb{U}_n=\left\{\begin{pmatrix}1&&*\\&\ddots&\\0&&1\end{pmatrix}\right\}$. Equivalently G is isomorphic to a subgroup of $\mathbb{U}_n$.

- Connected subgroups $B\subset G$ solvable, $R(G)\lhd G$ solvable, $R_u(G)\lhd G$ unipotent are Borel resp. radical resp. unipotent radical if they are maximal wrt given properties. The two radicals are unique, Borel subgroups are unique up to conjugation. Clearly $R_u(G)\subset R(G)\subset B$, and we have $R_u(RG)=R_uG$, $R(B)=B$. $R(G)$ is $\bigcap$ of all Borels, $R_u(G)$ is $\bigcap$ of all $R_u(B)$s over B Borel.

Note a torus is soluble so every (maximal) torus in G is contained in some Borel — for any $T \subset G$ maximal torus there are finitely many Borels $B_1, \dots, B_n \subset G$ containing T, and $R(G) = \bigcap_i B_i$, $R_u(G) = \bigcap_i R_u(B_i)$.

• If G is a connected soluble group, then $R_u G = G_u$ is the largest unipotent subgroup of G, and G/G_u is a torus. Moreover the sequence $1\!\!1 \to G_u \to G \to G/G_u \to 1\!\!1$ splits and the splittings $G/G_u \to G$ correspond to maximal tori in G. More generally if G is any group, $T \subset G$ maximal torus and $B \subset G$ a Borel containing T, then have $B = B_u \rtimes T$ so T is unique in B up to conjugation. It follows that T is unique in G up to conjugation, in fact (B,T) is unique in G up to conjugation.

• A connected group G is simple if it is noncommutative and has no nontrivial normal subgroups.
... almost-simple if noncommutative and has no nontrivial connected normal subgroups.
... semisimple if $R(G) = 1\!\!1$, ie it has no nontrivial connected normal subgroup that is soluble (equivalently commutative.) Equivalently G is reductive with finite center.
If G is semisimple and H is a minimal nontrivial normal subgroup then H is simple. There are finitely many such H, say $H_1, \dots, H_n$, they commute, $H_1 \cap \dots \cap H_n$ is finite, and $H_1 \times \dots \times H_n \to G$ is surjective.
... reductive if $R_u(G) = 1\!\!1$, ie it has no nontrivial normal subgroup that is unipotent.
Equivalently if every finite dimensional representation V is semisimple, ie. a (direct) sum of simple reps, equivalently completely reducible in the sense that every subrepresentation $W \subset V$ is split as $V = W \oplus V'$
Equivalently, some semisimple representation $G \to GL(V)$ has finite kernel. Equivalently, RG is a torus (follows from noting $RG = R_u G \rtimes T$ for some torus); then $RG \triangleleft ZG$ and the quotient ZG/RG is finite.
Note $[G,G]$ is semisimple, $[G,G] \cap RG$ is finite, $[G,G]$ commutes with RG and $G = [G,G] \cdot RG$. Conversely G is reductive if $\exists S, T \subset G$, S semisimple, T torus, $S \cap T$ finite, S, T commute, $G = ST$

• A group G is linearly reductive if its neutral component G^0 is reductive, equivalently every finite dimensional representation V is semisimple, equivalently completely reducible.

• The Lie algebra $\mathfrak{g}$ of an algebraic group G is its tangent space at identity — viewing G as a functor of points into the category of groups, have $\mathfrak{g} = \ker(G(k[\varepsilon]/(\varepsilon^2)) \to G(k)) \cong \mathrm{Hom}_k(I/I^2, k)$ where $I \subset k[G]$ is the ideal defining $\{e\} \subset G$. Then $\mathfrak{g}$ is naturally a k-vector space. Eg the Lie algebra of $GL(V)$ is $\mathrm{End}(V)$ when V is a finite dim vector space. Now taking derivatives gives a natural map $\mathrm{Aut}(G) \to GL(\mathfrak{g})$, composing this with the adjoint action $G \to \mathrm{Aut}(G)$; $g(h) = g^{-1}hg$ gives the adjoint representation $G \to GL(\mathfrak{g})$.

Taking derivatives, get the adjoint (Lie) representation $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, ie a map $[-,-]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. This defines the Lie bracket with its usual properties.

• A subalgebra is a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$, an ideal is a subspace with $[\mathfrak{h},\mathfrak{g}] \subset \mathfrak{h}$. If $H \subset G$ is a subgroup then its Lie algebra $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, if H is normal then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

• The center of G is the kernel of $\mathrm{Ad}: G \to \mathrm{Aut}(G)$; the center of $\mathfrak{g}$ is the kernel of $\mathrm{ad}: \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$ (ie all $x \in \mathfrak{g}$ such that $[x,-]=0$. If G is connected, $\mathfrak{z}(\mathfrak{g})$ is the Lie algebra of $Z(G)$. More generally suppose G is any group with subgroup H, write C_H for the centraliser of H and N_H for the normaliser of H. Let their Lie algebras be $\mathfrak{g}, \mathfrak{h}, \mathfrak{c}_H, \mathfrak{n}_H$. Then $\mathfrak{c}_H = \mathfrak{g}^H$ (H invariant part of $\mathfrak{g}$), and $\mathfrak{n}_H / \mathfrak{h} = (\mathfrak{g}/\mathfrak{h})^H$.

• A Lie algebra is solvable if it can be built from commutative ones by extensions, ie there is a finite sequence of subalgebras $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \dots \supset \mathfrak{g}_n = 0$ with each $\mathfrak{g}_{i+1}$ an ideal in $\mathfrak{g}_i$, and $\mathfrak{g}_i / \mathfrak{g}_{i+1}$ abelian. Can choose such that $\dim(\mathfrak{g}_i/\mathfrak{g}_{i+1}) = 1$. Equivalently $\mathfrak{g}$ is the Lie algebra of a connected solvable group.

... nilpotent if it can be built from commutative ones by central extensions, ie each $\mathfrak{g}_{i+1}$ above is in the center of $\mathfrak{g}_i$. Equivalently each $[x,-] \in \mathrm{End}\,\mathfrak{g}$ is nilpotent ($x \in \mathfrak{g}$), equivalently $\exists n > 0$ with $[x_1,[x_2,[\dots,x_n]]] = 0$ for all $x_1,\dots,x_n \in \mathfrak{g}$. Equivalently $\mathfrak{g}$ is the Lie algebra of a unipotent group.

• The Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is the largest solvable subalgebra, the radical $\mathfrak{r} \trianglelefteq \mathfrak{g}$ is the largest solvable ideal, the nilradical $\mathfrak{n} \trianglelefteq \mathfrak{g}$ is the largest nilpotent ideal. By Levi's theorem, the quotient $\mathfrak{g} \to \mathfrak{g}/\mathfrak{r}$ splits.

• A Lie algebra $\mathfrak{g}$ is simple if it is not abelian and has no nontrivial ideal.

... semisimple if the radical $\mathfrak{r}$ is trivial ie $\mathfrak{r} = 0$. Equivalently $\mathfrak{g}$ is reductive with trivial center. Equivalently, $\mathfrak{g} = \mathfrak{h}_1 \oplus \dots \oplus \mathfrak{h}_n$ where each $\mathfrak{h}_i \trianglelefteq \mathfrak{g}$ is a simple Lie algebra.

... reductive if radical $\mathfrak{r}$ is central ie $[\mathfrak{r},\mathfrak{g}] = 0$. Then $\mathfrak{r} = \mathfrak{z}(\mathfrak{g})$ and the Levi decomposition becomes $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ where $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ is semisimple and equals $[\mathfrak{g},\mathfrak{g}]$. Conversely $\mathfrak{g}$ is reductive iff $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$ for $\mathfrak{s}$ semisimple, $\mathfrak{a}$ abelian. Equivalently $\mathfrak{g}$ has a faithful representation V that is completely reducible (ie if $W \subset V$ is a subrep. then $V = W \oplus V'$). Can take V to be the adjoint rep $\mathfrak{g} \circlearrowright \mathfrak{g}$.

• If G is a Lie group and $T \subset G$ a maximal torus, then the centraliser $C_G(T) = \{g \in G \mid gt = tg \ \forall t \in T\}$ is a Cartan subgroup, this is connected and unique up to conjugation. If G is reductive then $T = C_G(T)$.

- For a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{t}\subset\mathfrak{g}$ is toral if it is abelian and each operator $\mathrm{ad}_t:\mathfrak{g}\to\mathfrak{g}$ $(t\in\mathfrak{t})$ is a diagonal matrix for some choice of basis. A subalgebra $\mathfrak{h}\subset\mathfrak{g}$ is Cartan if it centralises some maximal toral subalgebra, ie $\mathfrak{h}=\{x\in\mathfrak{g}\mid [x,t]=0\ \forall t\in\mathfrak{t}\}$. Equivalently $\mathfrak{h}$ is nilpotent and self-normalising ie $\mathfrak{h}=\{x\in\mathfrak{g}\mid [x,\mathfrak{h}]\subset\mathfrak{h}\}$. If $\mathfrak{g}$ is reductive then maximal toral algebras are self-centralising, so $\mathfrak{h}\subset\mathfrak{g}$ in this case is Carton if and only if it is maximal toral.

- Suppose G is semisimple and $T\subset G$ is a maximal torus, correspondingly we have the semisimple Lie algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}$. Now T is reductive and abelian so its simple representations are in bijection with its character lattice $X^*(T)=\mathrm{Hom}(T,\mathbb{G}_m)\subset\mathrm{Hom}(\mathfrak{h},k)=\mathfrak{h}^\vee$. The representation $T\hookrightarrow G\to GL(\mathfrak{g})$ must then split into a sum $\bigoplus_{\alpha\in X^*(T)}\mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is a $\oplus$ of simple representations given by α, ie $\forall t\in T$ and $y\in\mathfrak{g}_\alpha$, $ty=\alpha(t)y$ Equivalently, $\mathfrak{g}=\bigoplus_{\alpha\in\mathfrak{h}^\vee}\mathfrak{g}_\alpha$ is a simultaneous diagonalisation of the operators $\mathrm{ad}_x:\mathfrak{g}\to\mathfrak{g}$ $(x\in\mathfrak{h})$, so that for $y\in\mathfrak{g}_\alpha$ we have $[x,y]=\alpha(x)y$. Note then that $\mathfrak{g}_0$ is the centraliser of $\mathfrak{h}$ ie $\mathfrak{g}_0=\mathfrak{h}$. The non-zero αs which appear are the roots of $\mathfrak{g}$. Turns out this decomposition (hence the root system $\Phi\subset\mathfrak{h}^\vee$) determines $\mathfrak{g}$ completely, so determines G up to central isogeny: note $\mathfrak{h}$ is the dual to span Φ, each $\mathfrak{g}_\alpha$ $(\alpha\neq0)$ is 1-dimensional, $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]=\mathfrak{g}_{\alpha+\beta}$ if $\alpha,\beta,\alpha+\beta$ nonzero, $\mathfrak{g}_{2\alpha}=0$ for all roots, and $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha}$ is isomorphic to $\mathfrak{sl}_2=\begin{pmatrix}*&0\\0&*\end{pmatrix}\oplus\begin{pmatrix}0&*\\0&0\end{pmatrix}\oplus\begin{pmatrix}0&0\\*&0\end{pmatrix}$.

- The roots form a root system, ie they span $\mathfrak{h}^\vee$ and determine a complementary set of coroots $\Phi^\vee\subset\mathfrak{h}$ with a bijection $(-)^\vee:\Phi\to\Phi^\vee$ such that $\langle\Phi,\Phi^\vee\rangle\in\mathbb{Z}$, $\langle\alpha,\alpha^\vee\rangle=2\ \forall\alpha\in\Phi$, and the map $r_\alpha:\mathfrak{h}^\vee\to\mathfrak{h}^\vee$ given by $\beta\mapsto\beta-\langle\beta,\alpha^\vee\rangle\alpha$ is a reflection that maps Φ to Φ. The Weyl group $W=\langle r_\alpha\mid\alpha\in\Phi\rangle$ can be built from G,T: let $N_T\subset G$ be the normaliser of T, then N_T acts by conjugation on T, hence on $\mathfrak{h}$, and $T\subset N_T$ fixes T and $\mathfrak{h}$ pointwise so in fact $N_T/T\circlearrowright\mathfrak{h}$. Turns out $W\cong N_T/T$.

- Each Borel subgroup $B\subset G$ containing T (equivalently any Borel subalgebra $\mathfrak{b}\subset\mathfrak{g}$ containing $\mathfrak{h}$) determines a set of positive roots by $\alpha>0\Leftrightarrow\mathfrak{g}_\alpha\subset\mathfrak{b}$; this is a bijection between systems of positive roots and Borels containing T. Choice of Borel also determines positive coroots and dominant coweights.

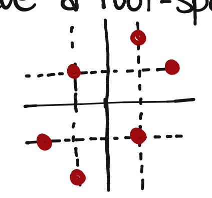
- Write $\Sigma\subset\mathfrak{h}$ for the $\mathbb{Z}$-span of Φ, this is the root lattice. Likewise $\Pi\subset\mathfrak{h}$, the $\mathbb{Z}$-span of $\Phi^\vee$, is the coroot lattice. The weight lattice $\Pi^\vee\subset\mathfrak{h}^\vee$ is the dual to Π, ie $\Pi^\vee=\{\beta\in\mathfrak{h}\mid\langle\alpha^\vee,\beta\rangle\in\mathbb{Z}\ \forall\alpha^\vee\in\Phi^\vee\}$. Then we have $\Sigma\subseteq X^*(T)\subseteq\Pi^\vee$, and $\Pi^\vee/X^*(T)$ is the fundamental group of G.

• Example: GL_3 over $\mathbb{C}$. The subgroup $\mathbb{U}_3 = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ is unipotent since all its elements are, so it is also nilpotent and soluble. The central series is $\mathbb{1} \subset \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset \mathbb{U}_3$. There is a larger soluble group $\mathbb{T}_3 = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$ containing $\mathbb{U}_3$, in fact is is Borel since $GL_3/\mathbb{T}_3$ is a proper flag variety (which can only happen if it contains a Borel subgroup). Now $\mathbb{U}_3$ is normal in $\mathbb{T}_3$ and the quotient is $\mathbb{G}_m^3$, with a splitting $\mathbb{G}_m^3 \cong \mathbb{D}_3 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$. It follows that $\mathbb{D}_3$ is a maximal torus in $\mathbb{T}_3$ (hence in GL_3) and $\mathbb{U}_3$ is the unipotent radical of $\mathbb{T}_3$. Likewise $\mathbb{U}_3^! = \begin{pmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{pmatrix}$ is the unipotent radical of $\mathbb{T}_3^! = \begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix}$. So the unipotent radical of GL_3, contained in $\mathbb{U}_3^! \cap \mathbb{U}_3$, must be the trivial group $\mathbb{1}$ ie GL_3 is reductive. The radical of GL_3 is thus contained in its center $\mathbb{G}_m \cdot \mathbb{1}$, and clearly this has no finite index subgroups so the radical of GL_3 is $\mathbb{G}_m \cdot \mathbb{1}$, the torus of scalar matrices. It follows that $[GL_3, GL_3] \cong SL_3$ is semisimple and every matrix in GL_3 is a product of a matrix in SL_3 with a scalar (its determinant).

The lie algebra $\mathfrak{gl}_3 = \mathrm{Mat}_3$ is thus also reductive, its radical (= its center) is the algebra $\mathfrak{r} = \mathbb{C} \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ of all scalar matrices (which commute with everything) and the semisimple part is the algebra $\mathfrak{sl}_3$ of traceless 3×3 matrices. Now choose the maximal torus $T = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \middle| \, abc = 1 \right\} \simeq \mathbb{G}_m^2 \subset SL_3$, with corresponding Cartan subalgebra $\mathfrak{h} = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \middle| \, a+b+c = 0 \right\} \subset \mathfrak{sl}_3$. Claim (guess) there is a root space decomposition

$$\mathfrak{sl}_3 = \mathfrak{h} \oplus \begin{pmatrix} \cdot & * & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & \cdot & * \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & * \\ \cdot & \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & \cdot & \cdot \\ * & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ * & \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & * & \cdot \end{pmatrix}.$$

Need to check T action on each component, say $M = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \in T$. Clearly $MXM^{-1} = X$ for all $X \in \mathfrak{h}$. If $X = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ then $MXM^{-1} = ab^{-1} \cdot X$, ie $\begin{pmatrix} \cdot & * & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ is a simple T-representation corresponding to the character ab^{-1}. Likewise the remaining are given by characters $ac^{-1} = a^2 b$, $bc^{-1} = ab^2$, ba^{-1}, $ca^{-1} = a^{-2}b^{-1}$, $cb^{-1} = a^{-1}b^{-2}$. So we do have a root-space decomposition! The character lattice $X^*(T) \simeq \mathbb{Z}^{\oplus 2}$ records the exponents of a, b so the root system is



Choosing the Borel $\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$ gives a set of positive roots $\alpha_0 = ab^{-1}$, $\alpha_1 = ab^2$, $\alpha_0 + \alpha_1 = a^2 b$.

Sanity check: $\begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in (\mathfrak{sl}_3)_{\alpha_0}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \in (\mathfrak{sl}_3)_{\alpha_1}$, $\left[\begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in (\mathfrak{sl}_3)_{\alpha_0 + \alpha_1}$.

What's the Weyl group? The torus T is normalised by $N_T = \left\{ \begin{pmatrix} x & \cdot & \cdot \\ \cdot & y & \cdot \\ \cdot & \cdot & z \end{pmatrix}, \begin{pmatrix} x & \cdot & \cdot \\ \cdot & \cdot & y \\ \cdot & z & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & x & \cdot \\ y & \cdot & \cdot \\ \cdot & \cdot & z \end{pmatrix}, \begin{pmatrix} \cdot & x & \cdot \\ \cdot & \cdot & y \\ z & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & x \\ y & \cdot & \cdot \\ \cdot & z & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & x \\ \cdot & y & \cdot \\ z & \cdot & \cdot \end{pmatrix} \right\}$ subject to $xyz = 1$, of course. To see the action $N_T \circlearrowright T$, compute eg that $\begin{pmatrix} x & \cdot & \cdot \\ \cdot & \cdot & y \\ \cdot & z & \cdot \end{pmatrix} \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \begin{pmatrix} x & \cdot & \cdot \\ \cdot & \cdot & y \\ \cdot & z & \cdot \end{pmatrix}^{-1} = \begin{pmatrix} a & & \\ & c & \\ & & b \end{pmatrix}$ ie each class of N_T/T, represented by a permutation matrix, acts on T by permuting diagonal entries. Thus $W = \mathfrak{S}_3$.

How does W act on $\mathfrak{h}$? Same computation on $X^*(T) \subset \mathfrak{h}^\vee$ (the inclusion measures the derivative of the character, eg $a \in X^*(T)$ gets mapped to $da \in \mathfrak{h}^\vee$ given by $da \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} = x$, since $a \begin{pmatrix} u+\varepsilon x & & \\ & v+\varepsilon y & \\ & & w+\varepsilon z \end{pmatrix} = u + \varepsilon x$) so eg the action of $M = \begin{pmatrix} * & \cdot & \cdot \\ \cdot & \cdot & * \\ \cdot & * & \cdot \end{pmatrix}$ is $M \cdot da = da \circ M = da$, $M \cdot db = db \circ M = dc = -da - db$, ie $M(\alpha_0) = M(da - db) = 2da + db = \alpha_0 + \alpha_1$ likewise M sends $\alpha_1 \mapsto -\alpha_1$ ie $M \circlearrowright \mathfrak{h}^\vee$ by reflection in α_1.