(Olivier Schiffmann) Cohomological Hall Algebras.

§ Motivation — Representation theory of Kac–Moody algebras.

Given Γ a generalised Dynkin diagram, get a Kac Moody Lie algebra $\mathfrak{g}$. Choosing an orientation of Γ (we assume simply laced) we also get a quiver $Q=(I,E)$. Old and classical theorems relate $\mathfrak{g}$ to the representation theory of Q.

Recall that $\mathfrak{g}_\Gamma$ is given by generators and relations determined by Γ. In particular there is a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$. The adjoint action $\mathfrak{h}\circlearrowright\mathfrak{g}$ gives a decomposition $\mathfrak{g}=\bigoplus_{\alpha\in\mathfrak{h}^*_{\mathbb{R}}}\mathfrak{g}_\alpha$ into weight spaces, where $\mathfrak{h}^*_{\mathbb{R}}=\mathrm{Hom}(\mathfrak{h},\mathbb{R})$ and for $x\in\mathfrak{h}$, $y\in\mathfrak{g}_\alpha$ we have $[x,y]=\alpha(x)y$. This is such that $\mathfrak{g}_0=\mathfrak{h}$. Say $\alpha\neq0$ is a <u>root</u> if $\mathfrak{g}_\alpha\neq0$, write $\Delta\subset\mathfrak{h}^*_{\mathbb{R}}$ for the set of roots. Have a decomposition $\Delta=\Delta^{re}\amalg\Delta^{im}$ into real and imaginary roots. Have $\dim\mathfrak{g}_\alpha=1$ if $\alpha\in\Delta^{re}$. The Lie algebra is graded; if $\alpha,\beta\in\Delta$ then $\alpha+\beta\in\Delta$ and $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subset\mathfrak{g}_{\alpha+\beta}$.

$Q_{\mathfrak{g}}\subset\mathfrak{h}^*_{\mathbb{R}}$, the $\mathbb{Z}$-span of Δ, is the root lattice. This has a Cartan form $\langle -,-\rangle$ and for any choice of simple roots $\Pi=\{\alpha_i\}_{i\in I}$ get positive halves $\Delta^+\subset\Delta$, $n^+=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$. The enveloping algebra of n^+ has <u>Chevalley generators</u> $\langle e_i \mid i\in I\rangle$.

On the quiver side fix a field $\mathbb{k}$ and consider the category $\mathrm{Rep}_{\mathbb{k}}Q$. This is heredetary, and the numerical Grothendieck group $K_0^{num}(\mathrm{Rep}_{\mathbb{k}}Q)$ is freely generated by vertex simples. This has a symmetric Euler form $[M,N]=\hom(M,N)-\mathrm{ext}^1(M,N)+\hom(N,M)-\mathrm{ext}^1(N,M)$.

From this data we define $\Delta_Q^+:=\{d\in\mathbb{N}^I \mid \exists$ indecomposable $M\in\mathrm{Rep}_{\mathbb{k}}Q$ with $\underline{\dim}(M)=d\}$. This decomposes as $\Delta_Q^+=\Delta_Q^{+,re}\amalg\Delta_Q^{+,im}$ where $d\in\Delta_Q^{+,re}\Leftrightarrow$ some (equivalently all) indecomposable M of dimension d are rigid (ie $\mathrm{ext}^1(M,M)=0$).

Theorem [Gabriel, Kac]. For any choice of positive roots, there is a lattice isomorphism $K_0^{num}\mathrm{Rep}_{\mathbb{k}}Q\simeq Q_{\mathfrak{g}}$ sending $[S_i]$ to the simple positive root α_i, and this maps Δ_Q^+ $(\Delta_Q^{+,re},\Delta_Q^{+,im})$ isomorphically to Δ^+ (resp $\Delta^{+,re},\Delta^{+,im}$). Furthermore $\forall d\in\mathbb{N}^I$ $\exists A_{Q,d}(t)\in\mathbb{Z}[t]$ such that for any prime power q, $A_{Q,d}(q)$ is precisely the number of absolutely indecomposable d-dimensional $\mathbb{F}_q$-representations of Q (up to iso).

This is Level 0 — a statement about K_0, so homological dimension 0 (semisimplification).
On level 1, we consider Hall Algebras or constructible $\overline{\mathbb{F}_q}$-sheaves and recover $U_q(n)$.

On level 2, taking the CoHA of $T^*\mathrm{Rep}\,Q$ recovers the Yangian — a looped version of $U_q(\mathfrak{n})$.
On level 3, the critical CoHA of $T^*[-1](T^*\mathrm{Rep}_k Q)$. In practice this endows more structure onto level 2.

§ Level 1: Hall Algebras

Fix a finite field $\Bbbk = \mathbb{F}_q$, and let $\mathcal{M}_{Q,\underline{d}}(\Bbbk)$ be the groupoid of $\underline{d}$-dimensional $\Bbbk$-representations of Q. This has finitely many isomorphism classes. Define $H_{Q/\Bbbk} := \bigoplus_{\underline{d}} \mathrm{Fun}(\mathcal{M}_{Q,\underline{d}}(\Bbbk), \mathbb{C})$, and equip this with a binary operation as follows — $\mathcal{M}^{ext}_{Q,d_1,d_2}(\Bbbk)$ is the groupoid of short exact sequences $0\to M\to N\to R\to 0$ of $\Bbbk Q$-representations with $\dim M = d_1$, $\dim R = d_2$. This has maps $\mathcal{M}_{Q,d_1}(\Bbbk) \times \mathcal{M}_{Q,d_2}(\Bbbk) \xleftarrow{\varpi} \mathcal{M}^{ext}_{Q,d_1,d_2}(\Bbbk) \xrightarrow{\pi} \mathcal{M}_{Q,d_1+d_2}(\Bbbk)$
The fiber of q over (M,R) is the groupoid of all extensions $0\to M\to N\to R\to 0$, so looks like $\text{"}\frac{\mathrm{Ext}^1(R,M)}{\mathrm{Hom}(R,R)\times\mathrm{Hom}(M,M)}\text{"}$
while the fiber of p counts d_1-dimensional subrepresentations of N, ie looks like a Grassmannian.
Writing $H_d := \mathrm{Fun}(\mathcal{M}_{Q,d}(\Bbbk), \mathbb{C})$, get a multiplication $m: H_{d_1}\otimes H_{d_2} \longrightarrow H_{d_1+d_2}$; $(f,g)\mapsto q^{\frac{1}{2}\langle d_1,d_2\rangle}\pi_*\varpi^!(f\otimes g)$
In other words, $(f*g)(N) = m(f,g)(N) = q^{\frac{1}{2}\langle d_1,d_2\rangle}\sum_{M\subseteq N} f(N/M)g(M)$

There exists a similar formula for a comultiplication. Consider the natural pairing $\langle f,f'\rangle = \text{"}\int_{\mathcal{M}_d} f\cdot f' d\mu\text{"}$, ie $\langle f,f'\rangle = \sum_M \frac{f(M)\,f'(M)}{|\mathrm{Aut}\,M|}$. The multiplication and comultiplication are adjoint with respect to $\langle -,-\rangle$.

Theorem [Ringel, Green]. The multiplication and comultiplication make H into a (twisted) self dual bialgebra

Note that by construction, $\mathrm{grdim}\, H_Q := \sum_d (\dim H_d) z^d = \prod_d \frac{1}{(1-z^d)^{\#\text{indec. reps of dim } d}} = \mathrm{Exp}_{t,z}\left(\sum_d A_{Q,d}(t)z^d\right)\Big|_{t=q}$
Here $\mathrm{Exp}_{t,z}$ is the plethystic exponential that handles the discrepancy of counts between absolutely indecomposable and indecomposable representations, by counting Galois orbits.
Eg if V is a $\mathbb{N}^\ell$-graded vector space then $\sum_{\underline{n}} \mathrm{grdim}_{\underline{n}}(\mathrm{Sym}^\bullet V) z^{\underline{n}} = \mathrm{Exp}_z\left(\sum_{\underline{n}} \mathrm{grdim}_{\underline{n}}(V) z^{\underline{n}}\right)$.

What is the structure of H_Q over $\mathbb{F}_q$? First look at the spherical Hall algebra H_Q^{sph} which is the subalgebra of H_Q generated by indicator (characteristic) functions of vertex simples $\mathbb{1}_{S_i}$.

Theorem [Ringel, Green]. H_Q^{sph} is isomorphic to $U_{q^{1/2}}(\mathfrak{n}^+_{\mathfrak{g}})$ as a quantum group.

Proof idea. It is easy to show the existence of a bialgebra morphism $U_{q^{1/2}}(\mathfrak{n}^+) \longrightarrow H_Q^{sph}$. Now $U_{q^{1/2}}(\mathfrak{n}^+)$ has a compatible non-degenerate Hopf pairing, the Drinfeld pairing, making the kernel of this map an ideal and coideal which does not contain a Chevalley generator. This implies the kernel is trivial.

Theorem (Hausel)(Kac's conjecture 1). $\mathrm{grdim}\ H^{sph}_{Q/\mathbb{F}_q} = \mathrm{Exp}\left(\sum_d A_{Q,d}(0) z^d\right)$. Equivalently the constant term of the Kac Polynomial $A_{Q,d}(t)$ is the dimension of the weight space $\mathfrak{g}_{\underline{d}}$.

It is then natural to ask whether there is a grading on H_Q such that the graded dimension recovers the entire Kac polynomial? In this situation, is there a graded Lie algebra $\tilde{\mathfrak{g}}$ such that $A_{Q,d}(t)$ gives the graded dimension of $\tilde{\mathfrak{g}}_{\underline{d}}$ (with $\mathfrak{g}$ = degree 0 part of $\tilde{\mathfrak{g}}$) ?

§ Level 1': The geometry of Hall algebras

Work over $\mathbb{k} = \overline{\mathbb{F}_q}$ or $\mathbb{C}$. Now the groupoid $\mathcal{M}_{Q,d}(\mathbb{F}_q)$ considered above gives the $\mathbb{F}_q$ points of an Artin stack $\mathcal{M}_{Q,d}$ with $\mathbb{k}$-points $\left[\bigoplus_{e\in E} \mathrm{Hom}(\mathbb{k}^{d_{s(e)}}, \mathbb{k}^{d_{t(e)}}) / \prod_i GL(d_i, \mathbb{k})\right]$. This is a smooth global quotient. The faisceaux-fonctions allows us to replace $H_{Q/\mathbb{F}_q}$ with $D^b_{cons}(\mathcal{M}_Q)$ (with coefficients in $\overline{\mathbb{Q}}_\ell$, possibly). We lift all structures to this stack.

Note now $\mathcal{M}_{Q,d_1} \times \mathcal{M}_{Q,d_2} \xleftarrow{\varpi} \mathcal{M}^{ext}_{Q,d_1,d_2} \xrightarrow{\pi} \mathcal{M}_{Q,d_1+d_2}$ is a diagram of smooth stacks. The ext-stack, which parametrises short exact sequences, is the total space of $R\mathrm{Hom}_Q(\mathcal{E}_{d_1}, \mathcal{E}_{d_2})[1]$ where $\mathcal{E}_d \in \mathrm{Coh}(\mathcal{M}_{Q,d} \times I)$ is the tautological bundle. Since $\mathrm{Rep}\, Q$ is heredetary, $R\mathrm{Hom}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2})[1]$ is of perfect amplitude $[0,1]$ ie ϖ is a stack vector bundle. Picture: two vector bundles $E_{-1}, E_0 \to \mathcal{M}_{Q,d_1} \times \mathcal{M}_{Q,d_2}$ and a map $E_{-1} \to E_0$. Taking the quotient E_0/E_{-1} (where $E_{-1} \circlearrowright E_0$ by the given map), get $\mathcal{M}^{ext}$ a vector bundle of rank $\langle d_1, d_2\rangle = \hom(d_1,d_2) - \mathrm{ext}^1(d_1,d_2)$. The map ϖ is not representable but it is almost like a vector bundle.

The map π has fibers given by Quot schemes, so it is proper.

We could just take $D^b_{cons}(\amalg_d \mathcal{M}_{Q,d})$ with some binary operation $\pi_* \circ \varpi^!$ but this would be too large to do much with. Instead following Lusztig, we consider the subcategory $\mathcal{Q}_Q \subset D^b_{cons}(\mathcal{M}_Q)$ generated by the constant sheaves $\overline{\mathbb{Q}}_\ell|_{\mathcal{M}_{Q,\alpha_i}}$ $(i\in I)$ (Most of the time $\mathcal{M}_{Q,\alpha_i}$ is $[*/\mathbb{G}_m]$), and require that $\mathcal{Q}_Q$ be stable under shifts, $\oplus$, direct summands, and $\mathrm{Ind} := \pi_* \varpi^*$.

Note ϖ is smooth and π is proper so $\mathrm{Ind}(\overline{\mathbb{Q}}_\ell|_{\mathcal{M}_{d_1}} \boxtimes \overline{\mathbb{Q}}_\ell|_{\mathcal{M}_{d_2}})$ will be a $\oplus$ of shifts of simple perverse sheaves, the same remains true for iterated Ind of $\overline{\mathbb{Q}}_\ell|_{\mathcal{M}_{d_1}} \boxtimes \cdots \boxtimes \overline{\mathbb{Q}}_\ell|_{\mathcal{M}_{d_s}}$. The category $\mathcal{Q}_Q$ must contain each summand.

Write $\mathcal{P}_{\underline{d}}$ for the set of all simple perverse sheaves that lie in $\mathcal{Q}_{Q,\underline{d}} = \mathcal{Q}_Q \cap D^b_{cons}(\mathcal{M}_{Q,\underline{d}})$.
For $\mathcal{K} = "K_0^{gr}(\mathcal{Q}_Q)" := \bigoplus_d \bigoplus_{\mathbb{P} \in \mathcal{P}_d} b_{\mathbb{P}}[v,v^{-1}]$. The subspace $b_{\mathbb{P}} v^{\ell}$ is generated by the class of $\mathbb{P}[\ell]$.

Get an associative product $\mathcal{K}^{\otimes 2} \to \mathcal{K}$ induced by Ind.

Let $\mathcal{K}' \subset \mathcal{K}$ be the subalgebra generated by classes of $\overline{\mathbb{Q}}_\ell|_{\mathcal{M}_{\alpha_i}}$, seems reasonable to define since taking direct summands makes $\mathcal{Q}_Q$ very large and we want to get an analog of the spherical Hall algebra.

Theorem [Lusztig]. We have $\mathcal{K} = \mathcal{K}'$ (!!). Further there is a coproduct $\mathcal{Q}_Q \to \mathcal{Q}_Q^{\boxtimes 2}$ given by hyperbolic localisation, and a twisted bialgebra isomorphism $U_{v^2}(\mathfrak{n}) \xrightarrow{\sim} \mathcal{K}'$ sending $E_i \mapsto v^{-1} b_{[\overline{\mathbb{Q}}_\ell|_{\mathcal{M}_{\alpha_i}}]}$

Corollary: The preimages of $b_{\mathbb{P}}$ ($\mathbb{P} \in \mathcal{P}$) give a basis of $U_{v^2}(\mathfrak{n})$, the so-called canonical basis.

... Olivier talks about the proof. One key step in proving $\mathcal{K}' = \mathcal{K}$ is an induction by removing source vertices of Q. What if there are no sources? Lusztig related $\mathcal{M}_Q$ to $\mathcal{M}_{Q'}$, where Q' is obtained from Q by flipping one edge.

Moral: it is best to work with all orientations of Γ, where any two orientations Q, Q' have an associated Fourier transform $\mathcal{Q}_Q \to \mathcal{Q}_{Q'}$. In fact it is better to work with $T^*\mathcal{M}_Q$ which accounts for all orientations simultaneously.

General principle: If X is smooth, there is a characteristic cycle map $CC: K_0 D^b_{cons}(X, \mathbb{Q}) \to Lag^{\mathbb{G}_m}(T^*X)$ associating each class to a Lagrangian $\mathbb{G}_m$-equivariant cycle in T^*X. The map satisfies

$CC(\mathcal{L} \text{ local system}) = (-1)^{\dim X} rk(\mathcal{L}) \cdot \underset{\uparrow \text{zero section}}{[X]}$; $CC(\mathbb{P} \text{ simple perverse sheaf}) \subset pr^{-1}(\operatorname{Supp} \mathbb{P})$

Turns out $T^*\mathcal{M}_{Q,d}$ has a Lagrangian substack Λ_d (Lusztig Lagrangian) such that the characteristic cycles map $CC: \mathcal{K}|_{v=-1} = K_0(\mathcal{Q}_Q) \longrightarrow Lag^{\mathbb{G}_m}(T^*\mathcal{M}_Q)$ is an isomorphism onto $H_{top}(\Lambda)$. Evidently this means $\sum \dim(H_{top} \Lambda_d) z^d = \operatorname{Exp}(\sum A_{Q,d}(0) z^d)$; but we can do one better and recover the complete Kac polynomial by taking the total homology as $\sum \operatorname{grdim}(H_\bullet \Lambda_d) z^d = \operatorname{Exp}(\sum_d A_{Q,d}(t) z^d)$.

Both have algebra structures; this is the Cohomological Hall Algebra.

§ Level 2: Cohomological Hall algebras

We begin by constructing $T^*\mathrm{Rep}_d Q$. Recall $\mathrm{Rep}_d Q = E_d / G_d$ where $E_d = \bigoplus_{e\in E} \mathrm{Hom}(\mathbb{k}^{d_{s(e)}}, \mathbb{k}^{d_{t(e)}})$ on which $G_d = \prod_i GL(d_i, \mathbb{k})$ acts. The tangent complex at a point x is $\mathbb{T}_x(\mathrm{Rep}_d Q) = \mathfrak{g}_d \xrightarrow{a} E_d$ where $\mathfrak{g}_d = \mathrm{Lie}(G_d)$, $a: (y_i) \mapsto (x_e y_{s(e)} - y_{t(e)} x_e)$ is the infinitesimal action. To make sense of this, note x can be given by a point $(x_e) \in E_d$ indexed over edges (e) of Q.

Consequently the cotangent complex at x is $\mathbb{T}^\vee_x(\mathrm{Rep}_d Q) = E_d^\vee \xrightarrow{a^\vee} \mathfrak{g}_d^\vee$. But writing $Q^* = (I, E^*)$ for the opposite quiver, we can identify $E_d^\vee$ with $E_d^* = \bigoplus_{e\in E} \mathrm{Hom}(\mathbb{k}^{d_{t(e)}}, \mathbb{k}^{d_{s(e)}})$. The trace map allows us to identify $\mathfrak{g}_d^\vee \simeq \mathfrak{g}_d$, then we have $a^\vee(y_e^*) = \sum_e [y_e^*, x_e] \in \mathfrak{g}_d$ (sum of commutators).

Putting this together, $T^*\mathrm{Rep}_d Q = \text{Total Space}(E_d^* \to \mathfrak{g}_d^\vee) = \mathrm{Tot}_{BG_d}(T^*E_d \xrightarrow{\mu} \mathfrak{g}_d^\vee)$.
moment map, given by $a^\vee$.

The tangent complex $\mathbb{T}(T^*\mathrm{Rep}_d Q)$ is $\mathfrak{g}_d \to T^*E_d \to \mathfrak{g}_d^\vee$ (in degrees $-1, 0, 1$). The piece in degree 1 is a reflection of the derived stacky structure. The (underived) classical truncation is the kernel of the map $T^*E_d \to \mathfrak{g}_d^\vee$, ie $\mu^{-1}(0)/G_d$.

The classical truncation is a singular stack, but it is evidently cut by the zero section of a vector bundle so it is a "quasismooth derived stack".

Example. Take $Q = \bullet\circlearrowleft$ the Jordan quiver. Then $\mathrm{Rep}_d Q = \mathfrak{gl}_d / GL_d$, so that the moment map $T^*(\mathfrak{gl}_d) \xrightarrow{\mu} \mathfrak{gl}_d^\vee$ takes commutators. Consequently the cl. truncation is $cl(T^*\mathrm{Rep}_d Q) = \{(x,y) \in \mathfrak{gl}_d^2 \mid xy = yx\}/GL_d$.
This is well known to be irreducible.

Example. Take $Q = \bullet \rightrightarrows \bullet$ with $d = (1,1)$. Then $\mathrm{Rep}_d Q = \mathbb{A}^2/\mathbb{G}_m \times B\mathbb{G}_m$ (since the diagonal $\mathbb{G}_m \subset G_d$ acts trivially). Then $cl(T^*\mathrm{Rep}_d Q) = \{xy + x'y' = 0\}/\mathbb{G}_m \times B\mathbb{G}_m$.

We now introduce the Lusztig Lagrangian, which records the singular supports of all of Lusztig's perverse sheaves. This generically has no derived structure, but at its singular points we do not know which derived enhancement makes it Lagrangian. So we stick to the classical picture.

The Lusztig Lagrangian is then the classical Lagrangian substack $\Lambda = \left\{ (x_e, x_e^*)_{e\in E} \;\middle|\; \begin{array}{l} \exists \text{ a restricted flag of } I\text{-graded} \\ \text{vector spaces } L_\bullet \text{ such that } \forall k, \\ x_e(L_k) \subseteq L_{k-1} \text{ but } x_e^*(L_k) \subseteq L_k \end{array} \right\}$

For the Jordan quiver, $\Lambda_d = \{(x,y) \in \mathfrak{gl}_d^2 \mid \substack{xy=yx,\\ x \text{ or } y \text{ nilpotent}}\} = \coprod_{\substack{\mathbb{O} \text{ orbit in}\\ \text{nilp cone } \mathcal{N}_{\mathfrak{gl}_d}}} T^*_{\mathbb{O}}$. Thus irreducible components of Λ_d are in bijection with nilpotent orbits.

Here is another perspective: $\mathrm{Rep}_d Q = \mathrm{Coh}_{0,d}(\mathbb{A}^1)$ parametrises length d coherent sheaves on $\mathbb{A}^1$, while $T^*\mathrm{Rep}_d Q$ is $\mathrm{Coh}_{0,d}(\mathbb{A}^2)$. Then $\Lambda_d \subset T^*\mathrm{Rep}_d Q$ parametrises length d coherent sheaves on $\mathbb{A}^2$ but with support in $\mathbb{A}^1 \times \{0\}$.

For $Q = \bullet \rightrightarrows \bullet$, $d=(1,1)$ we have $\mathrm{Rep}_d Q = \mathbb{A}^2/\mathbb{G}_m \times B\mathbb{G}_m = (\mathbb{P}^1 \cup B\mathbb{G}_m) \times B\mathbb{G}_m$. The nilpotency condition forces either both forward maps or both backward maps to vanish, so $\Lambda \cong \underset{\text{Closure of } \mathbb{P}^1}{\mathrm{Rep}_d Q} \sqcup \underset{\mathbb{A}^2\text{-bundle over } B\mathbb{G}_m}{\mathrm{Rep}_d^* Q}$ has two irreducible components.

<u>Theorem</u>. ① Λ is (generically) Lagrangian.

② The characteristic cycle map $cc: K|_{v=-1} \to \mathrm{Lag}^{\mathbb{G}_m} \mathcal{M}_Q$ is an isomorphism onto $H_{top} \Lambda = \bigoplus_{\substack{c \subset \Lambda \\ \text{irr comp}}} \mathbb{C}\cdot[c]$.

[① is due to Lusztig, ② due to Kashiwara and Y. Saito].

To prove both statements, an inductive structure on Λ ("Kashiwara Crystal Structure") is used.

<u>Corollary</u>. The dimension $\sum_d \dim H_{top}(\Lambda_d) z^d$ is the Plethystic exponential $\mathrm{Exp}(\sum_d A_{Q,d}(0) z^d)$.

<u>Question</u> Computation of Poincaré polynomials $P(\Lambda_d, t)$ and $P(T^*\mathrm{Rep}_d Q, t)$.

<u>Strategy</u>: Kac's polynomials compute volumes of certain $\mathbb{F}_q$-stacks so to relate the above Poincaré polynomials to Kac's polynomials we use cohomological purity results and point counts over $\mathbb{F}_q$.

<u>Remark</u>. For every edge $e \in E$ there is an associated $\mathbb{G}_m \circlearrowright \mathrm{Rep}_d Q$ which rescales the edge. Thus a torus $T = \mathbb{G}_m^{|E|+1}$ acts on $T^*\mathrm{Rep}_d Q$. It is a good idea to work T-equivariantly.

<u>Theorem</u> ① $H^T_\bullet(\Lambda, \mathbb{C})$ is pure, even, and free as an $H^\bullet_T$-module.

② $H^T_\bullet(T^*\mathrm{Rep}_d Q)$ is pure, even, and free as an $H^\bullet_T$-module

[① is Schiffmann-Vasserot, ② is more general and due to Ben Davison.]

Roughly, the idea for ① was to leverage analogous results for Nakajima quiver varieties. For ②, which holds more generally for 2CY categories, the idea is to relate $H_\bullet(T^*\mathrm{Rep}_d Q)$ to the critical homology of the moduli stack of objects in a 3CY categories (this is called dimension reduction). Then we again employ dimension reduction to relate this critical homology to H_* of a stack parametrising pairs $(x \in \mathrm{Rep}\,Q,\ \varphi \in \mathrm{End}(x))$. This stack is stratified by the eigenvalues of φ and everything reduces to the nilpotent case. In the Jordan stratification, all strata can be expressed in terms of $\mathrm{Rep}_d Q$.

We give a flavour of point counts now.

Theorem Over $\Bbbk = \mathbb{F}_q$, ① $\sum_d |T^*\mathrm{Rep}_d Q(\mathbb{F}_q)| \cdot q^{\langle d,d\rangle} z^d = \mathrm{Exp}_{t,z}\left(\frac{t}{t-1} \sum_d A_{Q,d}(t) z^d\right)\Big|_{t=q}$.

② $\sum_d |\Lambda_d(\mathbb{F}_q)|\, q^{\langle d,d\rangle} z^d = \mathrm{Exp}_{t,z}\left(\frac{1}{1-t^{-1}} \sum_d A^{nilp}_{Q,d}(t^{-1}) z^d\right)\Big|_{t=q}$

[① is Mozgovoy, ② is Schiffmann–Vasserot–Vaissek].

Proof (of ①) $T^*\mathrm{Rep}_d Q$ can be identified with $\{(M \in E_d,\ \varphi \in \mathrm{Ext}^1(M,M)^*)\}/G_d$. Thus we have

$$\sum_d |T^*\mathrm{Rep}_d Q(\mathbb{F}_q)|\, q^{\langle d,d\rangle} z^d = \sum_{\substack{\text{iso class}\\ M}} \frac{|\mathrm{Ext}^1(M,M)|}{|\mathrm{Aut}(M)|}\, q^{\langle M,M\rangle} z^{\dim M}.$$

By definition, $-\langle M,M\rangle = \dim \mathrm{Ext}^1(M,M) - \dim \mathrm{End}(M)$. Thus $|\mathrm{Ext}^1(M,M)| = q^{-\langle M,M\rangle} \cdot |\mathrm{End}(M)|$, ie the sum is reduced to $\sum_M {|\mathrm{End}(M)|}/{|\mathrm{Aut}(M)|} \cdot z^{\dim M}$. The quantity $|\mathrm{End}\,M|/|\mathrm{Aut}\,M|$ can be computed from the decomposition of M into indecomposables, so the sum becomes $\mathrm{Exp}\left(\sum_d \frac{t}{t-1} \cdot \#^{\text{abs indec reps}}_{\text{of dim } d} \cdot z^d\right)$
The computation is routine once familiarity with Plethystic exponentials is given. □

Combined with purity results, we then have an estimate on the size of $H_\bullet(T^*\mathrm{Rep}_d Q)$.

We can now define the (2D) Cohomological Hall algebra. It is convenient to phrase things using the preprojective algebra: to the quiver Q we associate the doubled quiver $\overline{Q} = (I, E \cup E^*)$ and take the quotient $\Pi_Q := \mathbb{C}\overline{Q} / \sum_e [e,e^*] = 0$. Thus $T^*\mathrm{Rep}_d Q = \mathrm{Rep}_d \Pi_Q$.

The algebra is then induced by the induction diagram $\mathrm{Rep}_{d_1}\Pi_Q \times \mathrm{Rep}_{d_2}\Pi_Q \xleftarrow{\varpi} \mathrm{Rep}^{ext}_{d_1,d_2} \xrightarrow{\pi} \mathrm{Rep}_{d_1+d_2}\Pi_Q$.
The map π is still some Quot scheme (ie is proper), and ϖ, while not smooth, can be pulled along.
'Why?' $\mathrm{Rep}^{ext}_{d_1,d_2}$, with derived structure, is the total space of $R\mathrm{Hom}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2})[1]$ and the map ϖ "is"
$\begin{pmatrix} R\mathrm{Hom}(\mathcal{E}_{d_2},\mathcal{E}_{d_2})[1] & R\mathrm{Hom}(\mathcal{E}_{d_1},\mathcal{E}_{d_2})[1] \\ 0 & R\mathrm{Hom}(\mathcal{E}_{d_1},\mathcal{E}_{d_1})[1] \end{pmatrix}$, a complex over $\mathrm{Rep}_{d_1}\Pi_Q \times \mathrm{Rep}_{d_2}\Pi_Q$. Thus ϖ is the total space of a three-term complex. Then there is a natural pullback $H_\bullet X \to H_{\bullet + 2\dim f} Y$ associated to any smooth map $X \xrightarrow{f} Y$ [see Adeel Khan]; concretely we represent $R\mathrm{Hom}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2})[1]$ as a three term complex

of vector bundles $\mathcal{E}_{-1} \to \mathcal{E}_0 \xrightarrow{b} \mathcal{E}_1$. The classical truncation is the total space of $\mathcal{E}_{-1} \to \mathrm{Ker}(b)$.

Then without appealing to derived structures, we pull back by considering

$$\begin{tikzcd} \mathrm{Tot}(\mathcal{E}_{-1} \to \mathcal{E}_0) \arrow[d, "\varpi_1"', "\text{smooth}"] & \mathrm{Tot}(\mathcal{E}_{-1} \to \mathrm{Ker}\, b) \arrow[l, "\varpi_2"', "\text{zero sec}^{n}\text{ of vector bundle}"] \\ \mathrm{Rep}_{d_1}\Pi_Q \times \mathrm{Rep}_{d_2}\Pi_Q & \end{tikzcd}$$

so ϖ_1^* and $\varpi_2^!$ exist ($\varpi_2^!$ is refined Gysin pullback).

<u>Theorem</u> [Schiffmann-Vasserot, Yang-Zhao] $\left(\bigoplus_d H_{\bullet - \langle d,d \rangle} \mathrm{Rep}\,\Pi_Q,\ \pi_* \varpi^!\right)$ is a graded associative algebra and ditto for $H_\bullet(\Lambda)$. Further, this holds torus-equivariantly and $H_\bullet^T(\mathrm{Rep}\,\Pi_Q)$ is an $H_T^\bullet$-algebra.

We now review some things we know about these algebras.

<u>We know</u>: size of $H_\bullet^T(\Lambda)$, $H_\bullet^T(\mathrm{Rep}_d \Pi_Q)$ is given by Kac's polynomials.
Also, $H_{top}(\Lambda) \subset H_\bullet^T(\Lambda)$, which is independent of T and corresponds to constant terms of Kac's polynomials, is (as a vector space) the enveloping algebra of a Kac-Moody algebra.

<u>Theorem</u> [Heinnecart] The map $cc: K^{gr}|_{v=-1} \xrightarrow{\sim} H_{top}\Lambda$ is an algebra isomorphism.

Describing the $H_T^\bullet$-algebra structure is hard. If we do the easier thing and localise to $\mathrm{Frac}(H_T^\bullet)$, we see that $H_\bullet^T(\Lambda_Q) \hookrightarrow H_\bullet^T(\mathrm{Rep}\,\Pi_Q)$ becomes an isomorphism. We expect this algebra to be a deformation of an extension $\tilde{\mathfrak{g}}[u]$ of $\mathfrak{g}[u]$. Such things are usually called "Yangians."

<u>Theorem</u> ① $H_\bullet^T(\Lambda)_{loc} \simeq H_\bullet^T(\mathrm{Rep}\,\Pi_Q)_{loc}$ is generated by $H_\bullet^T(\Lambda_\alpha)$ for $\alpha \in \{\alpha_i\}$.

② The map $H_\bullet^T(\Lambda_Q)_{loc} \to H_\bullet^T(\mathrm{Rep}\,\overline{Q})_{loc}$ identifies the domain with a small shuffle algebra.

③ There exists a presentation of $H_\bullet^T(\Lambda)_{loc}$ by generators and quadratic and cubic relations & Wheel conditions.

④ If Q is finite/affine, $H_\bullet^T(\Lambda)_{loc}$ is the usual/affine Yangian.

[① is Schiffmann-Vasserot with Negut if Q has edge loops, ③ is Negut-Sala-Schiffmann].

Things look nicer if we take only a subtorus which preserves the symplectic form (eg trivial T).

<u>Theorem</u> [Davison–Hennecart–Schlegel-Mejia, Davison–Hennecart–Kinjo, Jindal] $H_\bullet^T(\mathrm{Rep}\,\Pi_Q)$, if T preserves the symplectic form, is isomorphic to $U(\hat{\mathfrak{g}}_T^{BPS}) = \bigoplus_d \hat{\mathfrak{g}}_{d,T}^{BPS}$ where the affinized BPS Lie algebra $\hat{\mathfrak{g}}_{d,T}^{BPS}$ is isomorphic as a vector space to $\mathfrak{g}_{d,T}[u]$. Moreover, $\bigoplus \mathfrak{g}_{T,d}$ is a subalgebra (BPS Lie algebra), and this is a Borcherds algebra. Lastly, when $T=1$, this shows $\mathrm{grdim}\, \mathfrak{g}_d^{BPS} = A_{Q,d}(t^{-1})\, t^{\langle d,d\rangle}$.

The intersection homology of $(\mathrm{Rep}\,\Pi_Q)^{\mathrm{coarse}}$ is recorded as the Cartan matrix of $\bigoplus \mathfrak{g}_{T,d}$.

<u>Remark</u> In general, $H_\bullet^T(\mathrm{Rep}\,\Pi_Q) \otimes \langle \text{Tautological Classes} \rangle \otimes H_T^\bullet(\Lambda_Q)^{op}$ is isomorphic to a Maulik–Okounkov Yangian [Schiffmann–Vasserot, Botta–Davison]. Moreover $\mathfrak{g}^{BPS}$ gives the Maulik Okounkov Lie algebra.

§ From quivers to curves

Instead of $\mathrm{Rep}\,Q$, we can consider a smooth projective curve X and the category $\mathrm{Coh}\,X$. Then $T^*\mathrm{Rep}\,Q$ becomes the moduli stack of Higgs sheaves, while Λ_Q becomes the global nilpotent cone Λ_X. To spell this out, a Higgs sheaf is a sheaf on T^*X with proper support, ie a sheaf $\mathcal{F} \in \mathrm{Coh}\,X$ with a twisted endomorphism $\varphi: \mathcal{F} \to \mathcal{F} \otimes \omega_X$. Then Λ_X parametrises sheaves supported on the zero section, ie pairs $(\mathcal{F}, \varphi)$ with φ nilpotent.

Have $K_0 = \mathbb{Z}^{\oplus 2}$, $\langle -,- \rangle = \begin{pmatrix} 0 & 0 \\ 0 & 2-2g \end{pmatrix}$. Looks like affinization of $\bullet \circlearrowleft^{g}$ loops. There exists a "Kac polynomial" $A_{g,r,d}$; abs. indec. vector bundles of rank r, degree d are $A_{g,r,d}(\underbrace{\sigma_1, \dots, \sigma_{2g}}_{\text{Weil numbers}})$.

Can define the Hall algebra of $X_g/\mathbb{F}_q$ and the geometric Hall algebra $\mathcal{H}$.

The cohomological Hall algebra also exists and has a nilpotent version; it is isomorphic up to completion to $U(\bigoplus_{r,d} \hat{\mathfrak{g}}_{r,d}^{BPS})$. This is graded using Weil numbers, we don't even know the degree zero piece!
We know $U(\hat{\mathfrak{g}}^{BPS}[0]) = \bigoplus H_{top}(\Lambda_X)$.

§ Surfaces

Example. Over $\mathbb{C}$, for $Q = \bullet\circlearrowleft$, we have $\Pi_Q = \mathbb{C}[x,y]$ so that $T^*\mathrm{Rep}\,Q = \mathrm{Rep}\,\Pi_Q \simeq \mathrm{Coh}_0\,\mathbb{A}^2$. Thus the COHA $H_\bullet(T^*\mathrm{Rep}\,Q)$ is the COHA of zero dimensional sheaves on the surface $\mathbb{A}^2$.

This is specific to the situation. For general S, $\mathrm{Coh}\,S$ is not quasicompact or global quotient. The stack $\mathrm{Coh}_0 \mathbb{A}^2$ is quasicompact, the resulting COHA is a very small piece of the COHA of the full surface.

<u>Theorem</u> [Schiffmann–Vasserot, Davison] For $T=\mathbb{C}^{*2}$, $H^T_\bullet(\mathrm{Coh}_0 \mathbb{A}^2)$ is the positive half $Y^+_\hbar(\hat{\mathfrak{gl}}_1)$ of the "affine Yangian" of $\mathfrak{gl}_1$. This algebra is a deformation of $U(w_{1+\infty})$ where $w_{1+\infty}$ is the Lie algebra of non-constant differential operators on $\mathbb{A}^2$.

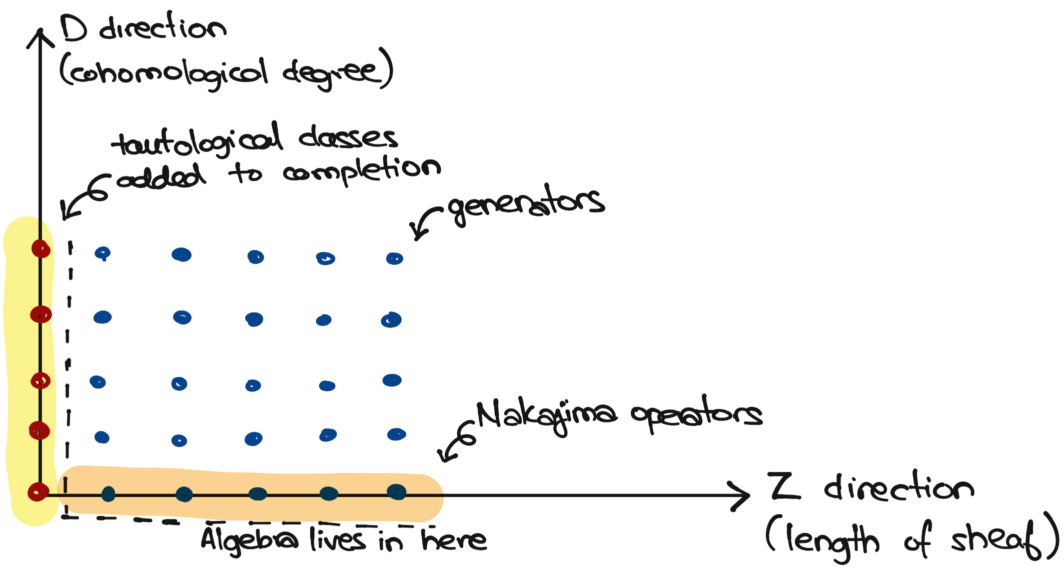
In the non-equivariant setting, $H_\bullet(\mathrm{Coh}_0 \mathbb{A}^2) = w_{1+\infty} = \langle D^m z^n \mid m\geq 0, n>0\rangle \big/ \; [D^m z^n, D^{m'} z^{n'}] = (mn'-m'n) D^{m+m'-1} z^{n+n'}$.

<u>Remark</u>. One can define a full Yangian $Y_\hbar(\hat{\mathfrak{gl}}_1) \simeq Y^+_\hbar(\hat{\mathfrak{gl}}_1) \otimes \mathbb{C}[\underbrace{c, p_1, p_2, \ldots}_{\text{tautological classes}}] \otimes Y^+_\hbar(\hat{\mathfrak{gl}}_1)^{op}$ satisfying

① $Y_\hbar(\hat{\mathfrak{gl}}_1) \circlearrowright H^T_\bullet(\mathcal{M}(r,\cdot))$ where the instanton spaces $\mathcal{M}(r,\cdot)$ generalise $\mathrm{Hilb}^0(\mathbb{C}^2) = \mathcal{M}(1,\cdot)$.

② $\left(Y_\hbar(\hat{\mathfrak{gl}}_1)\big|_{q=r}\right)^{\text{completion}} \twoheadrightarrow W(\hat{\mathfrak{gl}}_r) \circlearrowright H^T_\bullet(\mathcal{M}(r,\cdot))$ making the latter a Verma module for $W(\hat{\mathfrak{gl}}_r)$.

<u>Picture</u>.



Most of the above generalises to any smooth quasiprojective surface S. Write t_1, t_2 for its Chern roots, and write $K^c_0(S)_\mathbb{Q} = H^{even}_c(S,\mathbb{Q})$, $\delta = [\mathcal{O}_x]$ class of a point. The Chern roots are the "deformation parameters" that we cannot turn off. When they do have vanishing properties (ie if S is symplectic) we will recover undeformed Lie algebras.

For $\alpha \in K^c_0(S)_\mathbb{Q}$, write $\mathfrak{M}_\alpha = \{\mathcal{F} \in \mathrm{Coh}_{ps} S \mid [\mathcal{F}] = \alpha\}$. For example the classical truncation of $\mathfrak{M}_\delta$ is $S \times B\mathbb{G}_m$, while for $l>0$ there is a map $d(\mathfrak{M}_{l\delta}) \xrightarrow{\text{supp}} \mathrm{Sym}^l S$ whose fibers are $\prod_i \mathrm{Commuting}\,\mathrm{St}(\mathfrak{gl}_{d_i})$ (for some partition $l = \sum_i d_i$).

For $\alpha \notin \mathbb{N}\delta$, $\mathfrak{M}_\alpha$ is only locally of finite type. Eg if $S = T^*C$, $\alpha = r\cdot[C]$ we have that $\mathfrak{M}_\alpha$ is the stack of Higgs bundles of rank r (no imposed stability condition).

Writing $H(S)=\bigoplus_{\alpha} H_{\bullet}(\mathcal{M}_{\alpha},\mathbb{Q})$ (a topological vector space defined by taking the limit of homologies of qc open substacks of $\mathcal{M}_{\alpha}$). Inside $H(S)$ we have $H_0(S)=\bigoplus_{n\in\mathbb{N}} H_{\bullet}(\mathcal{M}_{n\delta},\mathbb{Q})$. Both are associative graded algebras [Sala–Schiffmann, Minets, Kapranov–Vasserot, Porta–Sala].

<u>Remark</u>. When S is not proper, it is useful to replace $H_{\bullet}(\mathcal{M}_{\ell\delta},\mathbb{Q})$ by $H^c_{\bullet}(\mathcal{M}_{\ell\delta},\mathbb{Q}):=H^{-\bullet}(\pi_!\,\mathrm{supp}_*(\mathbb{D}\mathbb{Q}_{\mathcal{M}_{\ell\delta}}))$ for $\pi:\mathrm{Sym}^{\ell}(S)\to pt$. Consequently for any open immersion $S\hookrightarrow S'$ we have algebra maps $i_!:H^c_0(S)\to H^c_0(S')$; $i^*:H_0(S')\to H_0(S)$.

$H^{(c)}_{\bullet}(\mathcal{M}_{\delta})\cong H^{\bullet}_{(c)}(S)[u]$, by mapping $x\in H^{\bullet}_{(c)}(S)[u]\mapsto x\cap[\mathcal{M}_{\delta}]^{red}$.

<u>Theorem</u> [Mellit–Minets–Schiffmann–Vasserot]. Assume S is cohomologically pure.

① The assignment $\underbrace{(\lambda u^n)\cap[\mathcal{M}_{\delta}]}_{\in H_{\bullet}(\mathcal{M}_{\delta})}\mapsto T_n(\lambda)$ gives an algebra isomorphism $H_0(S)\simeq W^+_{1+\infty}(S)$ where $W^+_{1+\infty}(S)$ has

$$T_n(a\lambda+b\mu)=aT_n(\lambda)+bT_n(\mu)\quad \forall a,b\in\mathbb{C},\ \lambda,\mu\in H^{\bullet}(S);$$

$$[T_m(\lambda\mu),T_n(\nu)]=[T_m(\lambda),T_n(\mu\nu)]$$

$$[T_m(\lambda),T_{n+3}(\mu)]-3[T_{m+1}(\lambda),T_{n+2}(\mu)]+3[T_{m+2}(\lambda),T_{n+1}(\mu)]-[T_{m+3}(\lambda),T_n(\mu)]-[T_m(\lambda),T_{n+1}(t_1t_2\mu)]$$
$$+\{T_m,T_n\}((t_1+t_2)\Delta_S\lambda\mu)=0$$

$$\sum_{w\in\mathrm{Sym}(3)} w\cdot[T_{m_3}(\lambda_3),[T_{m_2}(\lambda_2),T_{m_1+1}(\lambda_1)]]=0$$

② Assume $c_1=0$, $c_2=-q^2$ (S symplectic). Then $W^+_{1+\infty}(S)\simeq U(w^+_{1+\infty}(S))$, for $w^+_{1+\infty}(S)=\bigoplus_{\substack{m\geq 1\\ n\geq 0}} z^mD^n\lambda$ with relations $[z^mD^n\lambda,z^{m'}D^{n'}\mu]=z^{m+m'}((D+m'q)^nD^{n'}-D^n(D+mq)^{n'})\lambda\mu/q$.

Idea: $H_0(S)$ should act on the homology of interesting moduli spaces parametrising ≥ 1-dimensional sheaves. The general formalism, due to Kapranov–Vasserot, is that of a Hecke pattern.

<u>Definition</u> Assume S is proper for simplicity. A two-sided Hecke Pattern is a locally closed substack $X=\coprod X_{\alpha}\subset\mathcal{M}^{\geq 1}_S$ (stack of sheaves on S with no 0-dim subsheaf) such that X is stable under Hecke modifications, ie $\forall$ ses $0\to\mathcal{E}\to\mathcal{F}\to\mathcal{G}\to 0$ with $\mathcal{G}\in\mathcal{M}_{\mathbb{N}\delta}$, we have ($\mathcal{F}\in X\Rightarrow\mathcal{E}\in X$) and (($\mathcal{E}\in X$, $\mathcal{F}\in\mathcal{M}^{\geq 1}$) $\Rightarrow\mathcal{F}\in X$).

For example $\mathcal{M}^{\geq 2}$, $\mathcal{M}^{\geq 1}$, $\mathrm{Hilb}(S)$.

Proposition Let X be a two-sided Hecke pattern. $\forall \alpha \in K_0^c(S)$, $\mathbb{V}_\alpha = \bigoplus_{n\in\mathbb{Z}} H_\bullet(\mathcal{M}_{\alpha+n\delta} \cap X, \mathbb{Q})$ is a left (resp. right) module over $H_0(S)$.

Tautological classes. $H^\bullet_{taut}(X_\alpha, \mathbb{Q}) = \langle \int_S ch_i(\underbrace{\mathcal{E}_\alpha}_{\text{taut sheaf on } X_\alpha \times S}) \cup \lambda \mid i, \lambda \rangle$ There is a canonical morphism

$\Lambda(S) = \{$ free supercommutative algebra on symbols $\underline{ch_i}(\lambda) \mid i, \lambda \} \xrightarrow{ev_\alpha} H^\bullet_{taut}(X_\alpha)$.

The tautological homology $H_\bullet^{taut}(X_\alpha) = H^\bullet_{taut}(X_\alpha) \cap [X_\alpha]^{red.}$

Theorem Let X be a Hecke pattern. There is a map $\bigoplus_{n\in\mathbb{Z}} \Lambda(S) \overset{F}{\twoheadrightarrow} \mathbb{V}_{\alpha,taut} = \bigoplus_{n\in\mathbb{Z}} H_\bullet^{taut}(X_{\alpha+n\delta})$ (Fock Space)
which is an intertwining operator for the $H_0(S) = W^+_{1+\infty}(S)$ - action.

Ingredients : the main idea is to use Neguț's lemma to determine the action of length 1 Hecke operators and specialise to Hilbert scheme of points to perform explicit computations.

Remark This extends to an action of $W_{1+\infty}(S)$ and this recovers the Virasoro algebra.

Application (in progress). Assume S is symplectic, so $W^+_{1+\infty}(S) = U(w^+_{1+\infty}(S))$. There exists an affinized BPS lie algebra $\hat{\mathfrak{g}}^{BPS}$ associated to S such that $COHA(S) = U(\bigoplus_\alpha \hat{\mathfrak{g}}^{BPS}_\alpha)^{\text{completion}}$.

Theorem [Davison–Heinnecart–Kinjo–Schiffmann–Vasserot] Let α be a class curve such that $\alpha^2 > 0$.
① $\hat{\mathfrak{g}}^{BPS}_\alpha = H_\bullet^{taut}(\mathcal{M}_\alpha)$ (Markman's theorem for non co-prime case)
② $\exists$ explicit element $T_0(\beta)$ such that $ad(T_\delta\beta) : \hat{\mathfrak{g}}^{BPS}_\alpha \to \hat{\mathfrak{g}}^{BPS}_{\alpha+\delta}$ is an isomorphism (χ-independence).