What's on my rational curve?

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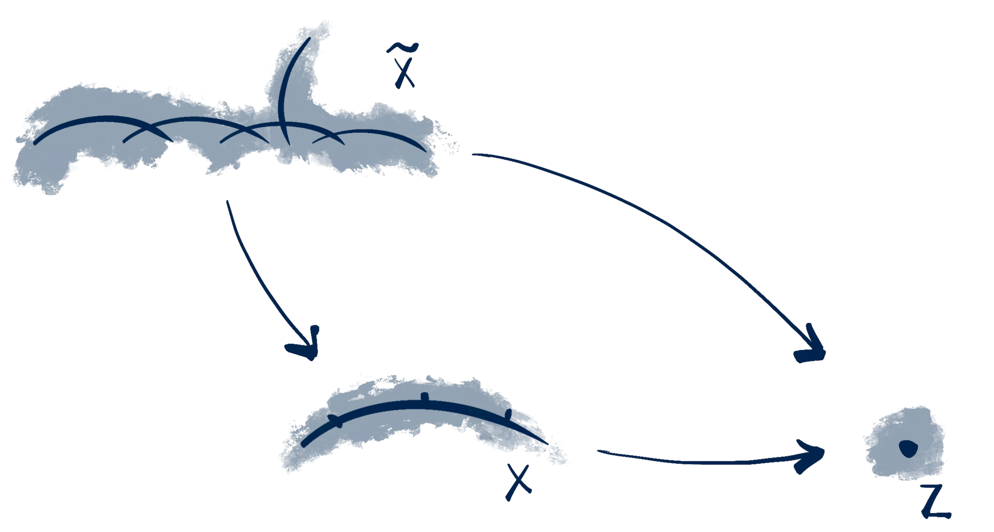
Theorem [Grothendieck–Birkhoff] Any complex $x \in \mathbf{D}^b\mathbb{P}^1$ can be uniquely expressed as a direct sum ($\oplus$) of suspended line bundles and torsion sheaves, so up to autoequivalences the only indecomposable objects are the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$ and fat skyscrapers $\mathcal{O}_{n\cdot p}$ at a closed point $p \in \mathbb{P}^1$.

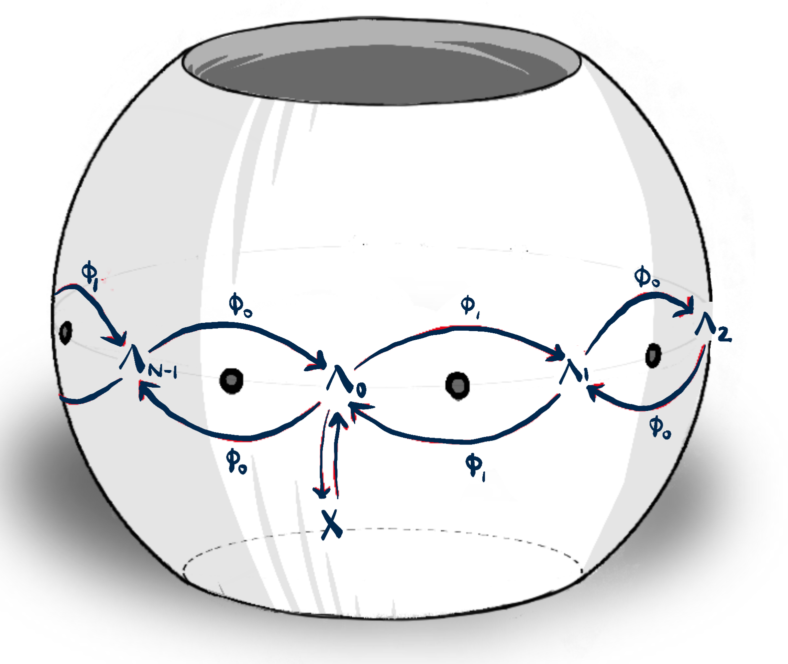
Given a crepant analytic neighbourhood X of a rational curve C, can we similarly classify complexes in $\mathbf{D}^b X$ up to autoequivalences?

Crepant analytic neighbourhood? Suppose $f : X \to Z$ is a birational map of surfaces with canonical singularities, such that $f^*\omega_Z = \omega_X$ and the connected rational curve C appears as an exceptional fiber.



Shrink to an analytic neighbourhood of the point $f(C)$, so that Z resembles a quotient singularity $\mathbb{C}^2/G$ for a finite group $G \subset SL(2,\mathbb{C})$. Then $X = f^{-1}(Z)$, a neighbourhood of C in the ambient surface, is a partial crepant resolution.

Up to what autoequivalences? It is natural to consider the action of $\mathbb{Z}$ (via suspension) and the Picard group (via tensor products), but there's more. Multiple tilting bundles on X give multiple algebras (say $\Lambda_0, \ldots, \Lambda_{N-1}$), derived-equivalent to X and to each other. When X is smooth, this recovers the McKay correspondence and spherical twist equivalences. This web of equivalences between $\mathbf{D}^b X$ and the $\mathbf{D}^b\Lambda_i$s can be systematically studied as an action of the *fundamental groupoid* of a hyperplane-complement in some affine space.



But why? Questions about spherical objects, stability manifolds, and autoequivalences in the McKay–setting have been long studied (Bridgeland, 2009; Ishii et al., 2010).

Understanding (spherical) objects in the derived category is essential to the study of the stability manifold of a K3 surface and compactifications thereof, not least for the study of string theory in low dimensions. From a purely algebro-geometric perspective such problems still emerge when classifying exceptional collections on Fano manifolds. When the surface has Picard rank greater than 1, such a classification must necessarily address spherical objects supported on embedded rational curves.

The fundamental group of a hyperplane-complement, an affine braid group, acts on $\mathbf{D}^b X$ by autoequivalences. Many group-theoretic questions about the braid group remain unanswered; considering $\mathbf{D}^b X$ and its associated invariants produces an array of algebraic and geometric objects on which the group naturally acts. Such techniques were used for instance to show that spherical braid groups are $K(\pi, 1)$ in August and Wemyss, 2022.

Simple complexes on a flopped curve[1]

The Calabi–Yau geometry of X demands we consider birational transformations when considering derived equivalences, these naturally arise out of wall-crossing constructions. The variety X can be recovered from Λ_0 as a moduli of stable representations, and changing the stability condition (or equivalently the algebra, see Wemyss, 2018) recovers alternate birational models of X which leave the locus away from C unchanged.

Suppose C is irreducible. This creates a situation analogous to a 3-fold flop, there are two birational models at play — X (associated to Λ_0) and W (associated to Λ_1).

Theorem If the complex $x \in \mathbf{D}^b X$ is supported on C and satisfies $\mathrm{Hom}^{<0}(x,x) = 0$, then up to standard equivalences, x is either

(1) a finite length Λ_i-module for some Λ_i, or
(2) a 2-term complex of coherent sheaves on X (or on W), or
(3) a direct sum of shifts of skyscraper sheaves on X (or on W).

If x additionally satisfies $\mathrm{Hom}(x,x) = \mathbb{C}$ (e.g. if x is spherical), then up to standard equivalences and Grothendieck duality, x is one of the thickened structure sheaves

$\mathcal{O}_C, \mathcal{O}_{2C}, \mathcal{O}_{3C}, \ldots, \mathcal{O}_{\ell C}$ (where ℓ is the length of the f-exceptional fiber),

or x is the unique non-split extension of $\mathcal{O}_{2C}$ by $\mathcal{O}_{3C}$ which exists when $\ell \geqslant 5$, or x is the structure sheaf of a closed point on X (or W).

We can then deduce global results about $\mathbf{D}^b_C X$, the full subcategory of complexes supported within C, by tracking simple and stable objects.

Corollary If K is the heart of an *algebraic* t-structure in $\mathbf{D}^b_C X$, then K is the track of some heart $\mathrm{fl}\,\mathit{mod}\,\Lambda_i$ under standard equivalences.

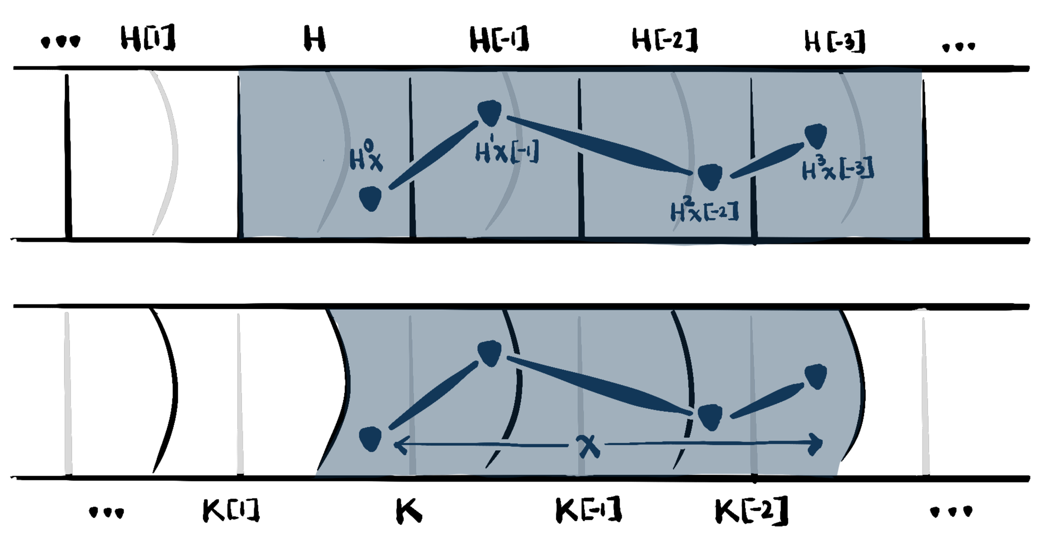
Corollary Each Bridgeland stability condition on $\mathbf{D}^b_C X$ has at most one accumulation point in its phase distribution, thus arises from an algebraic heart.

Corollary The stability manifold of $\mathbf{D}^b_C X$ is connected and contractible, and its projectivisation $\mathrm{Stab}(\mathbf{D}^b_C X)/\mathbb{C}$ is the universal cover of the *stringy Kähler moduli space* $\mathcal{M}_{SK}$ (a multi-punctured sphere, see Donovan and Wemyss, 2025). Thus the monodromy action of $\pi_1(\mathcal{M}_{SK})$ on $\mathbf{D}^b X$ (see Aspinwall, 2003) is faithful.

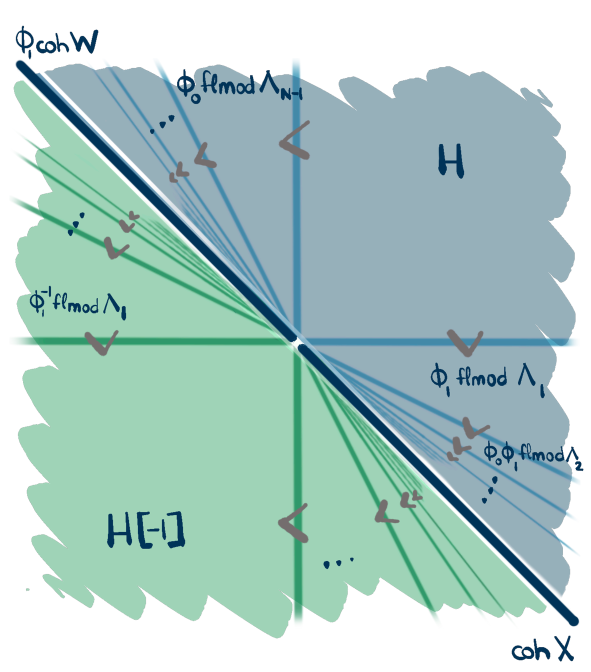
3-fold flops? The result holds also for partial resolutions of cDV singularities.

...via Torsion pairs and 3-fold flops[2]

To prove the classification result alongside, we attempt to inductively improve the *cohomological spread* of the complex $x \in \mathbf{D}^b X$ by HRS tilting — if x lies in a single cohomological degree with respect to $H = \mathrm{fl}\,\mathit{mod}\,\Lambda_0$, then x is a Λ_0-module and there is nothing to prove. Otherwise, the mild hypothesis on endomorphisms of x guarantees that H admits a tilt K with respect to which x has a smaller spread.



This led to the study and classification of tilts of H, by first systematically enumerating all the known tilts into a *heart fan* (see Broomhead et al., 2024), and then arguing that the affine geometry of the heart fan enables it to detect every tilt by showing any tilt is appropriately sandwiched between well-understood algebraic ones.



Theorem* Let $K \subset \mathbf{D}^b_C X$ be a heart obtained by tilting H. Then K is either

(1) the category of finite length modules $\mathrm{fl}\,\mathit{mod}\,\Lambda_i$ for some Λ_i (up to standard equivalences), or
(2) the category $\mathit{coh}\,X$ or $\Phi_1(\mathit{coh}\,W)$, or a tilt thereof in skyscrapers.

*An analogous result is also proved for multi-curve contractions.

Thus the complex x either lies in a tilt K of H (in which case we are done), or has a non-trivial (but strictly smaller) spread with respect to some tilt K. If some such K is $\Phi\,\mathrm{fl}\,\mathit{mod}\,\Lambda_i$, then replacing x by $\Phi^{-1}x$ allows the induction to continue. Otherwise all such K are geometric, which is only possible if the top and bottom cohomologies of x are skyscrapers, from whence we can deduce all cohomologies of x are so.

[1]Shimpi, P. (2025). Simple complexes on a flopped curve. *In preparation*.
[2]Shimpi, P. (2025). Torsion pairs and 3-fold flops. arXiv:2502.05146

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