Simple complexes on a flopped curve

Parth Shimpi

Abstract Studying crepant blow-ups of (compound) du Val singularities, we classify complexes of coherent sheaves which admit no negative self-extensions — such a complex, up to flops and mutation equivalences, must either be (1) a module over a derived–equivalent algebra, or (2) a two–term extension of a coherent sheaf by skyscraper sheaves, or (3) a direct sum of shifts of skyscrapers.

This translates into classifications of bricks, spherical objects, stability conditions, and algebraic t-structures in the local derived category; the lists populated by the homological minimal programme turn out exhaustive. We deduce that the Bridgeland stability manifold is connected, and that all basic tilting complexes on the variety (equivalently on the g−tame algebras derived-equivalent to it) are related by shifts and iterated mutation.

Proper birational morphisms $\pi : X \to Z$ that crepantly contract an irreducible rational curve $C \subset X$ to a point $\mathbf{0} \in Z$ have for decades piqued the curiosities of algebraic and symplectic geometers alike, and the advent of non-commutative and homological techniques in dimensions 2 and 3 [KV00; Bri02; VdB04; Wem18] has pushed into limelight the problem of understanding derived equivalences involving X.

Any solution to the problem must involve a classification of *spherical objects*, i.e. complexes which generate an autoequivalence of $\mathbf{D}^b X$ mirroring a Dehn twist. These are objects whose self-extensions resemble the singular cohomology of a sphere [ST01], and this condition is sufficiently restrictive that classification attempts have found success in specific cases where the geometry of X and its mirror is well-understood [IU05; KS24].

Walking the thin line that separates courage from naïveté, we forget the cohomological restriction and instead ask if it is possible to classify *all* complexes supported on C that admit no negative self-extensions.

Complexes, classified. Let $Z = \operatorname{Spec} R$ above be a complete local $\mathbb{C}$-scheme of dimension 2 (or 3), with an at worst canonical (resp. terminal) singularity at the closed point $\mathbf{0}$. Thus the map π models a partial resolution of a (compound) du Val singularity, obtained by blowing down all curves but one in some minimal model.

Certain wall–crossing rules (realising a flop when $\dim Z = 3$) yield another crepant partial resolution $W \to Z$ that is derived–equivalent to X. To study the pair of varieties, the homological minimal model programme [Wem18; DW25] furnishes a $(\mathbb{Z}/N\mathbb{Z})$-indexed family of *modification algebras* $\{\Lambda_0, \dots, \Lambda_{N-1}\}$ with equivalences

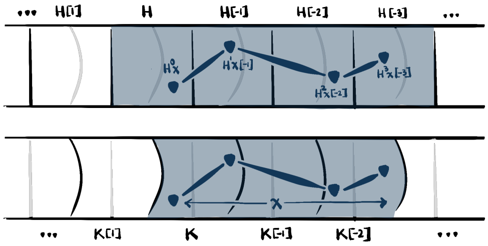
$$(*)\qquad \mathbf{D}^b X \xrightarrow{\text{VdB}} \mathbf{D}^b \Lambda_0, \qquad \mathbf{D}^b W \xrightarrow{\text{VdB}} \mathbf{D}^b \Lambda_{N-1}, \qquad \mathbf{D}^b \Lambda_i \underset{\Phi_i}{\overset{\Phi_i}{\rightleftarrows}} \mathbf{D}^b \Lambda_{i+1} \quad (i \in \mathbb{Z}/N\mathbb{Z}).$$

The number $N (\geqslant 1)$ of algebras involved is called the *helix period* associated to π, and depends only on the *length* ℓ of the exceptional fiber [see DW25, §2.1]. The *mutation functors* Φ_i, Φ_i are induced by (non-commutative deformations of) spherical objects, and provide a standard set of equivalences up to which we classify.

Theorem A (= 2.5, 2.6, 2.9). *If a complex $x \in \mathbf{D}^b X$ is supported on C and satisfies $\operatorname{Hom}^{<0}(x, x) = 0$, then up to a shift and iterated applications of the functors appearing in $(*)$ and their inverses, x is either*

- *(c1) a Λ_i-module for some $i \in \mathbb{Z}/N\mathbb{Z}$, or*
- *(c2) a direct sum of shifts of coherent sheaves with zero–dimensional support on X (or on W), or*
- *(c3) a coherent sheaf on X (or on W), or a two–term complex thereof determined as a Yoneda extension $x \in \operatorname{Ext}^2(\mathcal{G}, \mathcal{F})$ between coherent sheaves $\mathcal{F}, \mathcal{G}$ such that $\operatorname{Hom}(\mathcal{G}, \mathcal{F}) = 0$ and $\dim(\operatorname{Supp} \mathcal{G}) = 0$.*

Much like Hara–Wemyss' treatment [HW24] of analogous problems on the subcategory $\ker(\mathbf{R}\pi_*) \subset \mathbf{D}^b X$, we prove theorem A by attempting to inductively improve the 'spread' of a complex until it lies in a known heart. This begins by using its cohomologies with respect to $\mathsf{H} = \mathrm{fl}\,\mathit{mod}\,\Lambda_0$ to construct an intermediate heart K, such that the complex lies in strictly fewer degrees with respect to K than with respect to H (lemma 2.3). These are dimension–agnostic arguments that apply equally well to all settings — in particular, the same proof that applies to smooth surfaces also works for (possibly singular) 3-folds.



Much unlike Hara–Wemyss' category however, mutation alone may not suffice to reach the heart K. Indeed there are complexes (such as $\mathcal{O}_p \oplus \mathcal{O}_q[1]$ for $p, q \in X$) which can only be improved by tilting to a non-algebraic heart, and prima facie it is unclear how one continues the induction from here. By considering the complete lattice of *all* intermediate hearts [following Shi25] and exploiting the irreducibility of C, we obtain precise cohomological information about objects which do not improve by mutation (lemma 2.9), thus proving theorem A in § 2.

Bricks and spheres. A classification of spherical objects, and more generally semibricks (i.e. complexes whose self-extensions resemble those of simple modules) then follows from theorem A, the classification of semibrick Λ_0–modules [Shi25, theorem B], and Donovan–Wemyss' analysis of simple Λ_i-modules [DW25, §1.2].

Theorem B (= 3.1). *If a complex* $x \in \mathbf{D}^b X$ *supported on* C *satisfies* $\mathrm{Hom}^{<0}(x, x) = 0$ *and* $\mathrm{Hom}(x, x) = \mathbb{C}$, *then up shifts and iterated applications of the functors appearing in* $(*)$ *and their inverses,* x *is either*

(b1) a simple Λ_i*-module for some* $i \in \mathbb{Z}/N\mathbb{Z}$, *or*

(b2) a skyscraper sheaf at a closed point on X, *or on* W.

In fact if x *is described by (b1), then up to shifts, mutation functors, and the Grothendieck duality functor, it is one of the sheaves* $\mathcal{O}_C, \mathcal{O}_{2C}, \ldots, \mathcal{O}_{\ell C}$, *or the unique non-split extension* $\mathcal{Z}$ *of* $\mathcal{O}_{2C}$ *by* $\mathcal{O}_{3C}$ *which exists when* $\ell \geqslant 5$.

Evidently skyscraper sheaves at closed points are not spherical, i.e. any spherical object must arise from a simple Λ_i-module as in the above result. A direct computation [DW25, corollary 7.5] reveals that when X is a smooth 3-fold, a spherical object can only arise (as in theorem B) from the sheaves $\mathcal{O}_{\ell C}$ or $\mathcal{Z}$, and whether these are spherical depends on the specific geometry. For instance $\mathcal{Z}$ cannot be spherical unless $\ell = 5$, and $\mathcal{O}_{\ell C}$ is spherical if and only if the associated top Gopakumar–Vafa invariant is 1. In the special case when X is the conifold resolution (i.e. when $X \dashrightarrow W$ is the Atiyah flop), this provides a purely algebro-geometric proof of the fact that every spherical object is in the orbit of $\mathcal{O}_C$ under standard autoequivalences — a statement that was first proved by Keating–Smith [KS24, theorem 1.3] via symplectic dynamics on the mirror manifold.

Tilting complexes, **algebraic t-structures**, **and stability conditions**. Bricks detect Morita equivalences such as $(*)$ by picking out images of simple modules; traditionally such concerns are addressed via projectives instead by considering *tilting objects* — these are complexes $t \in \mathbf{D}^b X$ which satisfy $\mathrm{Hom}^i(t, t) = 0$ whenever $i \neq 0$ and generate the subcategory of perfects, thus inducing a derived equivalence between X and $\mathrm{End}(t)$ [Ric89].

Indeed Van den Bergh [VdB04] constructs the equivalence $\mathrm{VdB} : \mathbf{D}^b X \to \mathbf{D}^b \Lambda_0$ by identifying an indecomposable vector bundle $\mathcal{N} \in \mathcal{C}oh\, X$ with globally generated dual, such that the object $\mathcal{N} \oplus \mathcal{O}_X = \mathrm{VdB}^{-1}(\Lambda_0)$ is tilting. Replacing either summand with its left or right approximation by the other (i.e. *mutating*) constructs four more tilting objects, and hence four equivalences out of $\mathbf{D}^b \Lambda_0$ — these are the functors $\Phi_0^{\pm}, \Phi_{-1}^{\pm}$ in $(*)$. The remaining equivalences are likewise obtained by systematically propagating the mutation, thus enumerating a class of tilting objects that are *basic* (i.e. have non–isomorphic indecomposable summands) and sit in a tetravalent exchange graph. We show that there are no more such complexes.

Theorem C (= 3.2). *Every basic tilting complex in* $\mathbf{D}^b X$ *can be obtained from a shift of Van den Bergh's tilting bundle* $\mathcal{N} \oplus \mathcal{O}_X$ *by finitely many left or right mutations in indecomposable summands. Equivalently, every basic tilting complex of* Λ_i*–modules (*$i \in \mathbb{Z}/N\mathbb{Z}$*) can be obtained by iteratively mutating indecomposable summands of* $\Lambda_i[n]$ *for some* $n \in \mathbb{Z}$.

The classical tilting theory toolkit excels at populating a list of tilting objects out of a single such instance, it however generally falls short of identifying when such a list is exhaustive (notable exceptions are the cases of Dynkin preprojective algebras and Donovan–Wemyss' contraction algebras, where Aihara–Mizuno [AM17] and August [Aug20] leverage the finiteness of involved g–fans to obtain a complete classification.) Theorem C exhibits the first class of g–tame (in particular, silting–indiscrete) algebras for which the issue can be settled, namely two–vertex contracted affine preprojective algebras and certain modification algebras (algebras in these classes arise as Λ_is when $\dim Z = 2$ or 3 respectively.)

The proof is a *consequence* of a classification of Morita equivalences, more precisely we identify collections that could be the images of simple modules under the inclusion of a bounded heart $\mathit{mod}\Lambda \hookrightarrow \mathbf{D}^b\Lambda \xrightarrow{\sim} \mathbf{D}^b X$ for some module–finite R–algebra Λ. Any such R–linear equivalence (including the functors in $(*)$) restricts to the full subcategories supported on the singular point $\mathbf{0} \in Z$, i.e. the subcategories

$$\mathbf{D}^0 X = \{x \in \mathbf{D}^b X \mid \operatorname{Supp}(x) \subseteq C\}, \qquad \mathbf{D}^{\mathrm{fl}}\Lambda = \{x \in \mathbf{D}^b\Lambda \mid \text{each } H^i(x) \in \mathit{mod}\Lambda \text{ has finite length}\}.$$

The natural heart $\mathrm{fl}\mathit{mod}\,\Lambda = \mathbf{D}^{\mathrm{fl}}\Lambda \cap \mathit{mod}\Lambda$ is the extension–closure of its finitely many simples, whose images in $\mathbf{D}^0 X$ form a *simple–minded collection* which can be tracked and identified using theorem A.

Theorem D (= 3.1). *Given any bounded t-structure with algebraic (i.e. Artinian and Noetherian) heart $H \subseteq \mathbf{D}^0 X$, there is an index $i \in \mathbb{Z}/N\mathbb{Z}$ and equivalence $\mathbf{D}^0 X \to \mathbf{D}^{\mathrm{fl}}\Lambda_i$ composed of functors appearing in $(*)$ and their inverses that restricts to an equivalence between H and a shift of the full subcategory $\mathrm{fl}\mathit{mod}\,\Lambda_i$ of finite length Λ_i-modules.*

Another key consequence of theorem D is the connectedness of the manifold parametrising Bridgeland stability conditions on $\mathbf{D}^0 X$, an object of interest not least because it allows for an analysis of the autoequivalence group of $\mathbf{D}^b X$. Such analyses were carried out by Bridgeland [Bri09] and Hirano–Wemyss [HW23], where they found that all stability conditions arising from the standard hearts $\mathrm{fl}\mathit{mod}\,\Lambda_i \subset \mathbf{D}^{\mathrm{fl}}\Lambda_i$ $(i \in \mathbb{Z}/N\mathbb{Z})$ lie in a single connected component which is preserved by autoequivalences composed of the mutation functors $(*)$.

Then theorem D implies that a stability condition outside this component, if it exists, must only correspond to t-structures with non–algebraic hearts, in particular admitting no gaps in its phase distribution. By using theorem B to explicitly control which phases can be occupied by any collection of stable objects, we preclude the possibility of such a stability condition existing and deduce the following.

Theorem E (= 3.3). *Given any locally finite Bridgeland stability condition (Z, P) on $\mathbf{D}^0 X$, there is a phase $\varphi \in \mathbb{R}$ such that the heart $P(\varphi, \varphi+1]$ coincides with $\Phi(\mathrm{fl}\mathit{mod}\,\Lambda_i)$, for some index $i \in \mathbb{Z}/N\mathbb{Z}$ and an equivalence $\Phi : \mathbf{D}^b\Lambda_i \to \mathbf{D}^b X$ composed of the functors in $(*)$ and their inverses. In particular the stability manifold $\operatorname{Stab}(\mathbf{D}^0 X)$ is connected.*

Faithful monodromy on Kähler moduli. Viewing $\mathbf{D}^b X$ as the topological B-model of a superconformal field theory [see Bri06], one expects to recover a class of symmetries $G \subseteq \operatorname{Aut}(\mathbf{D}^b X)$ via variation of the complexified Kähler structure [Asp03, §4] on a conjectured moduli space modelled as a submanifold of $G \backslash \operatorname{Stab}(\mathbf{D}^0 X)/\mathbb{C}$.

This proposal was brought to fruition [Bri09; Tod08; HW23; DW25] by taking G to be the subgroup of R–linear Fourier–Mukai equivalences (that preserve a distinguished connected component of $\operatorname{Stab}(\mathbf{D}^0 X)$)[1] and computing that (said connected component of)[1] the double quotient $G\backslash \operatorname{Stab}(\mathbf{D}^0 X)/\mathbb{C}$ is an $(N+2)$-punctured sphere, a space reasonably declared to be the *stringy Kähler moduli space* $\mathcal{M}_{\mathrm{SK}}$.

The system of functors $(*)$ can then be seen as a representation of the fundamental groupoid $\Pi_1(\mathcal{M}_{\mathrm{SK}})$ with $N+2$ basepoints as in the figure below, with G arising from the fundamental group at $\mathbf{D}^b X$. Faithfulness of this representation is equivalent to simply–connectedness of $\operatorname{Stab}(\mathbf{D}^0 X)$, and is hence intertwined with long–standing questions on contractibility of stability manifolds.

[1] This clause is now redundant in light of theorem E.

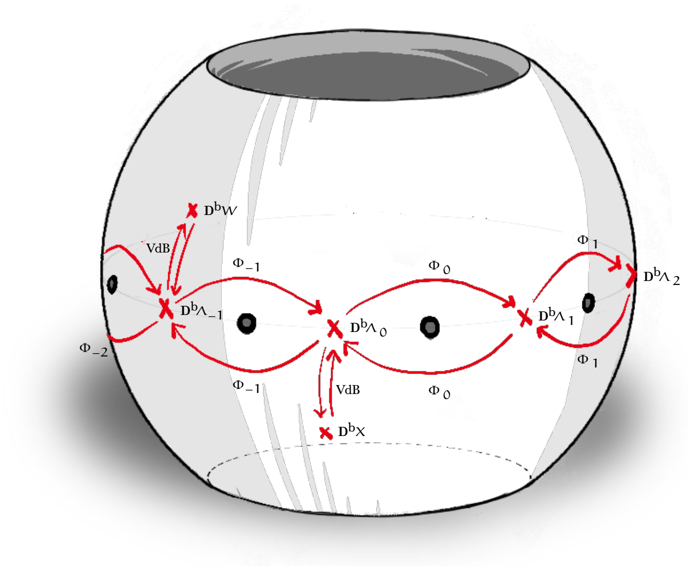


Figure 1. Labelling basepoints (marked ✗) with categories, and homotopy classes of paths with functors as shown gives a representation of the fundamental groupoid $\Pi_1(\mathcal{M}_{SK}, N+1)$.

Loops at a fixed basepoint then give autoequivalences of the label, for instance the anti-clockwise monodromy of $\mathbf{D}^b X$ around the north pole is given by the functor $\mathrm{VdB}^{-1} \circ \Phi_{-1} \circ \ldots \circ \Phi_0 \circ \mathrm{VdB}$, computed to coincide with the tensor–action of a generator of $\operatorname{Pic} X$. Likewise monodromy around the south pole recovers the action of $\operatorname{Pic} W$, and monodromies around the equatorial punctures correspond to twist functors in bricks described in theorem B (b1) [DW25, §6.3].

Both questions — of faithfulness and of contractibility — can then be settled using Roberts–Woolf's upcoming work [RW+], where they claim that each connected component of $\operatorname{Stab}(\mathcal{T})$ admits a contracting flow whenever $\mathcal{T}$ is a triangulated category with Grothendieck group $\mathbb{Z}^{\oplus 2}$. Theorem E, combined with Roberts–Woolf's claim, thus proves the folklore conjecture recorded below for convenience.

Conjecture F. *The stability manifold of* $\mathbf{D}^0 X$ *is homotopy equivalent to a point, and the quotient* $\operatorname{Stab}(\mathbf{D}^0 X)/\mathbb{C}$ *is the universal cover of* $\mathcal{M}_{SK}$. *Consequently the group* G *of* R*–linear Fourier–Mukai autoequivalences of* $\mathbf{D}^b X$ *is isomorphic to the fundamental group of* $\mathcal{M}_{SK}$*, i.e. a free group on* $N+1$ *generators.*

Acknowledgements. Many thanks to Michael Wemyss and Wahei Hara for generously sharing their insights which guided this project; to Jon Woolf and Stefan Roberts for explaining their work on stability spaces; and to Nick Rekuski and Timothy De Deyn for helpful comments.

Funding. The author was supported by ERC Consolidator Grant 101001227 (MMiMMa).

Open Access. For the purpose of open access, the author has applied a Creative Commons Attribution (CC:BY) licence to any Author Accepted Manuscript version arising from this submission.

§ 2 The taxonomy of objects

Let $\pi : X \to Z$ be the crepant contraction of an irreducible curve $C \subset X$ as in the introduction, and fix a complex $x \in \mathbf{D}^b X$ that is supported on C and satisfies $\operatorname{Hom}^i(x, x) := \operatorname{Hom}(x, x[i]) = 0$ whenever $i < 0$.

To prove theorem A, we pass across Van den Bergh's equivalence $\mathrm{VdB} = \mathbf{R}\operatorname{Hom}(\mathcal{N} \oplus \mathcal{O}_X, -)$ [see e.g. DW25, §2.2]. It is convenient to suppress VdB from the notation, making the identifications $\mathbf{D}^b X \simeq \mathbf{D}^b \Lambda_0$ and $\mathbf{D}^0 X \simeq \mathbf{D}^{fl} \Lambda_0$ and treating x as an object of either category. Thus working with respect to the standard heart $H = \mathrm{fl}\,\mathit{mod}\,\Lambda_0$ in $\mathbf{D}^{fl}\Lambda_0$, each cohomology object $H^i(x) \in H$ is a finite–length Λ_0–module, and x is an extension of the objects $\{H^i(x)[-i] \mid i \in \mathbb{Z}\}$. In particular, if $a \leqslant b$ are integers such that $H^i(x) = 0$ whenever $i \notin [-b, -a]$, then x lies in the full subcategory $H[a, b] = \langle H[b] \cup H[b-1] \cup \ldots \cup H[a] \rangle$, where $\langle - \rangle$ denotes the extension–closure.

Notation 2.1. We write $x \in H[\![a, b]$ to mean $x \in H[a, b]$ and $x \notin H[a+1, b]$ (equivalently, $x \in H[a, b]$ and $H^{-a}(x) \neq 0$). Other variations such as $x \in H[a, b]\!]$ and $x \in H[\![a, b]\!]$ are defined likewise. The notation naturally extends when measuring cohomologies with respect to other bounded hearts $K \subset \mathbf{D}^{fl}\Lambda_0$.

Now we may, up to replacing x with a shift thereof, assume that x lies in $H[\![0,n]\!]$ for some integer $n \geqslant 0$ (called the *spread* of x.) If the equality $n = 0$ held true then $x \in H$ would be a finite length Λ_0–module as described in theorem A (c1), so it suffices to only consider the case $n > 0$ and attempt to reduce the spread by updating H. Our primary tool to accomplish this is ***tilting in torsion pairs*** à la Happel–Reiten–Smalø [HRS96].

Lemma 2.2. *Suppose* $T \subseteq H$ *is a torsion class, corresponding to the torsion-free class* $F \subseteq H$ *and tilted heart* $K = \langle F[1] \cup T \rangle$.

(i) The torsion class T *contains the top cohomology* $H^0(x)$ *if and only if* x *lies in* $K[\![0,n]\!]$.

(ii) The torsion-free class F *contains the bottom cohomology* $H^{-n}(x)$ *if and only if* x *lies in* $K[\![-1,n-1]\!]$

Proof. By construction, K lies in $H[0,1]$ and hence H (in particular, each cohomology $H^{-i}(x)$) lies in $K[-1,0]$. Now x is an extension of the objects $H^{-i}(x)[i] \in K[i-1,i]$ for $i \in [0,n]$, of which all except $H^0(x)$ clearly lie in $K[0,n]$. Thus if $H^0(x)$ lies in $T \subseteq K$, then $H^0(x)$ (and hence also x itself) lies in $K[0,n]$. It is also clear that in this case x cannot lie in $K[1,n] \subset H[1,n+1]$, since $H^0(x) \neq 0$ by assumption. Hence x lies in $K[\![0,n]\!]$.

Conversely if x lies in $K[\![0,n]\!]$, then $\mathrm{Hom}(x,k) = 0$ for all $k \in K[<0]$ from whence it follows (e.g. by truncating to the coaisle of H) that $\mathrm{Hom}(H^0x,k) = 0$ for all $k \in K[<0]$. Thus $H^0(x)$ lies in $K[\geqslant 0]$, and hence in $T = K[\geqslant 0] \cap H$. This proves (i); the proof of (ii) is similar. □

The Abelian category H is Artinian and Noetherian, so the inclusion order on torsion–free (resp. torsion) classes in H forms a *complete lattice* $\mathrm{torf}(H)$ (resp. $\mathrm{tors}(H)$) which is (anti–)isomorphic to the poset of tilted hearts

$$\big[H, H[1]\big] = \big\{K \text{ heart of a bounded t-structure} \mid H \leqslant K \leqslant H[1]\big\},$$

where the latter is considered an interval in the set of all t-structures partially ordered by coaisle–containment [see e.g. Shi25, §2.2]. That is to say, we have naturally isomorphic partial orders $\mathrm{tors}(H)^{\mathrm{op}} \simeq \big[H, H[1]\big] \simeq \mathrm{torf}(H)$ that admit arbitrary suprema and infima. In particular there is a minimal torsion class containing $H^0(x)$ with corresponding tilted heart $\overline{K}$, and a minimal torsion–free class containing $H^{-n}(x)$ with corresponding tilted heart $\underline{K}$. Lemma 2.2 then characterises these hearts in terms of the partial order as

$$\big[H, \overline{K}\big] = \big\{K \in \big[H, H[1]\big] \mid x \in K[\![0,n]\!]\big\}, \qquad \big[\underline{K}, H[1]\big] = \big\{K \in \big[H, H[1]\big] \mid x \in K[\![-1,n-1]\!]\big\}.$$

Lemma 2.3. *We have the inequality* $\underline{K} \leqslant \overline{K}$. *In particular, the poset of t-structures on* $\mathbf{D}^0X$ *contains a* non-empty interval $\big[\underline{K}, \overline{K}\big]$ *containing hearts* K *such that* x *lies in* $K[\![0,n-1]\!]$.

Proof. The statement claims the containment of torsion classes $\overline{K} \cap H \subseteq \underline{K} \cap H$, hence it suffices to show that $H^0(x)$ is contained in $\underline{K} \cap H = {}^{\perp}(H^{-n}x)$. In other words, we must show $\mathrm{Hom}(H^0x, H^{-n}x) = 0$. But this readily follows from the hypothesis $\mathrm{Hom}(x, x[-n]) = 0$, by truncating first to the aisle and then to the coaisle and noting that the truncation functors are adjoint to inclusions (alternatively, by using the spectral sequence $E_2^{p,q} = \bigoplus_i \mathrm{Hom}^p(H^ix, H^{i+q}x) \Longrightarrow \mathrm{Hom}^{p+q}(x,x)$ to deduce that $\mathrm{Hom}(x,x[-n]) = \mathrm{Hom}(H^0x, H^{-n}x)$.) □

§ 2.1 Classifying two–term complexes. Lemma 2.3 asserts that the spread of x can *always* be improved by tilting once, in particular if x has spread $n = 1$ then it is an object in some tilted heart $K \in \big[H, H[1]\big]$. Such hearts are well–understood in terms of the functors (∗), we briefly discuss the constructions and classification.

The mutation functors can be defined from a $\mathbb{Z}$–indexed sequence of rigid reflexive R–modules $\{\ldots, V_{-1}, V_0, V_1, \ldots\}$ ordered such that V_{-1}, V_0 are pushforwards of $\mathcal{N}$, $\mathcal{O}_X$ respectively, and any three consecutive modules sit in an exchange sequence [see DW25, §2.3]. Writing $M_i = V_{i-1} \oplus V_i$, the modules $\mathrm{Hom}_R(M_i, M_{i+1}) \in \mathit{mod}(\mathrm{End}_R M_i)$ and $\mathrm{Hom}_R(M_{i+1}, M_i) \in \mathit{mod}(\mathrm{End}_R M_{i+1})$ are both tilting and induce R–linear equivalences

$$\mathbf{D}^b(\mathrm{End}_R M_i) \underset{\Psi_i := \mathbf{R}\mathrm{Hom}(\mathrm{Hom}_R(M_{i+1}, M_i), -)}{\overset{\Psi_i := \mathbf{R}\mathrm{Hom}(\mathrm{Hom}_R(M_i, M_{i+1}), -)}{\rightleftarrows}} \mathbf{D}^b(\mathrm{End}_R M_{i+1})$$

indexed over the mutated summand as in [DW25, §3.2]. Proposition 4.3 *ibid* then exhibits the inequalities of hearts $\Psi_i\, \mathrm{fl}\mathit{mod}(\mathrm{End}_R M_{i+1}) < \mathrm{fl}\mathit{mod}(\mathrm{End}_R M_i) < \Psi_i^{-1}\, \mathrm{fl}\mathit{mod}(\mathrm{End}_R M_{i+1})$ and further shows these relations are *covering* (i.e. the corresponding open intervals are empty.)

To recover the picture $(*)$ we note that the class group $\mathrm{Cl}(R) \cong \mathbb{Z}$ acts on the sequence (V_i), translating it in multiples of the helix period N. In particular we have canonical R–linear identifications $\beta_k : \mathrm{End}_R M_i \xrightarrow{\sim} \mathrm{End}_R M_{i+kN}$ for all $i, k \in \mathbb{Z}$ (see §3.3 *ibid*), so we declare $\Lambda_i = \mathrm{End}_R(M_i)$ and $\Phi_i = \Psi_i$ for $i \in \{0, \dots, N-1\}$ and observe that for any $k \in \mathbb{Z}$ there is an equality of functors $\Psi_{i+kN} = \beta_{-k} \circ \Phi_i \circ \beta_k$ [HW23, lemma 7.3]. Reading off t-structures in $\mathbf{D}^{\mathrm{fl}}\Lambda_0$ then recovers a part of the Hasse quiver of $[H, H[1]]$, given by

$$(!) \qquad \begin{array}{ccc} & \Phi_0(\mathrm{fl}\,\mathcal{mod}\,\Lambda_1)[1] \longrightarrow \Phi_0\Phi_1(\mathrm{fl}\,\mathcal{mod}\,\Lambda_2)[1] \to \cdots \to \Phi_{-1}^{-1}\Phi_{-2}^{-1}(\mathrm{fl}\,\mathcal{mod}\,\Lambda_{-2}) \to \Phi_{-1}^{-1}(\mathrm{fl}\,\mathcal{mod}\,\Lambda_{-1}) & \\ \nearrow & & \searrow \\ H[1] & & H \\ \searrow & & \nearrow \\ & \Phi_{-1}(\mathrm{fl}\,\mathcal{mod}\,\Lambda_{-1})[1] \to \Phi_{-1}\Phi_{-2}(\mathrm{fl}\,\mathcal{mod}\,\Lambda_{-2})[1] \to \cdots \to \Phi_0^{-1}\Phi_1^{-1}(\mathrm{fl}\,\mathcal{mod}\,\Lambda_2) \to \Phi_0^{-1}(\mathrm{fl}\,\mathcal{mod}\,\Lambda_1) & \end{array}$$

where the indices are modulo N. In [Shi25, theorem 5.13], numerical techniques are used to show that the two decreasing chains starting at $H[1]$ above have infima $\mathrm{VdB}(\mathcal{coh}\,X)[1]$ and $\Phi_0 \circ \mathrm{VdB}(\mathcal{coh}\,W)[1]$ respectively, so these geometric hearts too lie in $[H, H[1]]$. Further tilting the geometric hearts in torsion classes containing only skyscrapers and extensions thereof gives a class of 'mixed geometric hearts' (see §5.3 *ibid*) which are also tilts of H. The key result there is that this list is exhaustive and H has no other tilts, with a short proof (pertinent to the single–curve case we are in) sketched as [Shi25, Algorithm 1.1].

Theorem 2.4 [Shi25, Theorem A]. *Let* $K \in [H, H[1]]$ *be the heart of a bounded t-structure in* $\mathbf{D}^{\mathrm{fl}}\Lambda_0$*, obtained by tilting* $H = \mathrm{fl}\,\mathcal{mod}\,\Lambda_0$*. Then* K *admits one of the following descriptions.*

- *(h1) The heart* K *is the image of some standard heart* $\mathrm{fl}\,\mathcal{mod}\,\Lambda_i \subset \mathbf{D}^{\mathrm{fl}}\Lambda_i$ *(*$i = \mathbb{Z}/N\mathbb{Z}$*) under a functor* $\mathbf{D}^{\mathrm{fl}}\Lambda_i \to \mathbf{D}^{\mathrm{fl}}\Lambda_0$ *given by one of the composites* $[1] \circ (\Phi_{mN-1} \cdots \Phi_{i+1}\Phi_i)$*,* $[1] \circ (\Phi_{-mN} \cdots \Phi_{i-2}\Phi_{i-1})$*,* $(\Phi_{mN+i-1} \cdots \Phi_1\Phi_0)^{-1}$*, or* $(\Phi_{i-mN} \cdots \Phi_{-2}\Phi_{-1})^{-1}$ *(for some* $m \in \mathbb{N}$*).*
- *(h2) Identifying* $\mathbf{D}^{\mathrm{fl}}\Lambda_0 \simeq \mathbf{D}^0 X$ *across the equivalence* VdB*, the heart* K *is a tilt of the standard heart* $\mathcal{coh}\,X = \mathcal{Coh}\,X \cap \mathbf{D}^0 X$ *in a (possibly trivial) torsion class containing no sheaves with one–dimensional support. In other words, there is a subset* $U \subseteq X$ *such that* $K = \langle \{\mathcal{O}_p \mid p \in U\} \cup \{\mathcal{F}[1] \mid \mathcal{F} \in \mathcal{coh}\,X,\ \mathrm{Hom}(\mathcal{O}_p, \mathcal{F}) = 0 \text{ for all } p \in U\} \rangle$*.*
- *(h3) Identifying* $\mathbf{D}^{\mathrm{fl}}\Lambda_0 \simeq \mathbf{D}^0 W$ *across the equivalence* $\Phi_0 \circ \mathrm{VdB}$*, the heart* $K \subset \mathbf{D}^0 W$ *is a tilt of the standard heart* $\mathcal{coh}\,W$ *in a (possibly trivial) torsion class containing no sheaves with one–dimensional support. In other words, there is a subset* $U \subseteq W$ *such that* $K = \langle \{\mathcal{O}_p \mid p \in U\} \cup \{\mathcal{F}[1] \mid \mathcal{F} \in \mathcal{coh}\,W,\ \mathrm{Hom}(\mathcal{O}_p, \mathcal{F}) = 0 \text{ for all } p \in U\} \rangle$*.*

The following consequence is immediate.

Corollary 2.5. *If the complex* $x \in \mathbf{D}^{\mathrm{fl}}\Lambda_0$ *lies in* $H[0, 1]$*, then* x *adheres to one of the descriptions in theorem A.*

Proof. Lemma 2.3 guarantees that x lies in some heart K obtained as a tilt of $H = \mathrm{fl}\,\mathcal{mod}\,\Lambda_0$. If K admits the description (h1) in theorem 2.4, then $K = \Phi\,\mathrm{fl}\,\mathcal{mod}\,\Lambda_i$ for some $i \in \mathbb{Z}/N\mathbb{Z}$ and Φ a composite of shifts and mutation functors, hence $\Phi^{-1}(x) \in \mathrm{fl}\,\mathcal{mod}\,\Lambda_i$ is a Λ_i–module as in theorem A (c1).

Otherwise the heart K admits one of the descriptions (h2) or (h3), say (without loss of generality) the former. Thus K is a tilt of $\mathcal{coh}\,X$ in a torsion class $\langle \mathcal{O}_p \mid p \in U \rangle \subset \mathcal{coh}\,X$, determined by some subset $U \subset X$. It follows that the object $x \in K$ can be written as an extension $\mathcal{F}[1] \to x \to \mathcal{G} \to \mathcal{F}[2]$, where $\mathcal{F}$ and $\mathcal{G} \in \mathcal{coh}\,X$ are torsion–free (resp. torsion) with respect to the torsion pair relating K and $\mathcal{coh}\,X$. Thus $\mathcal{G} \in \langle \mathcal{O}_p \mid p \in U \rangle$ has zero–dimensional support, $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) = 0$, and x is determined by the morphism $\mathcal{G} \to \mathcal{F}[2]$ seen as an element of $\mathrm{Ext}^2(\mathcal{G}, \mathcal{F})$. $\square$

§ 2.2 Induction, dream scenario. Given $x \in H[\![0, n]\!]$ for $n \geqslant 2$, we may hope that the interval $[\underline{K}, \overline{K}] \subset [H, H[1]]$ contains a heart of the form ΦH for some *auto*equivalence $\Phi : \mathbf{D}^{\mathrm{fl}}\Lambda_0 \to \mathbf{D}^{\mathrm{fl}}\Lambda_0$ of desired form, so that replacing x by $\Phi^{-1}(x) \in H[\![0, n-1]\!]$ allows us to rinse and repeat until corollary 2.5 kicks in.

And we are correct in hoping so — the sub-poset of $[H, H[1]]$ containing only algebraic hearts is given by the Hasse quiver (!), so unless the window $[\underline{K}, \overline{K}]$ is exceptionally narrow it is bound to contain an image of $H = \mathrm{fl}\,\mathcal{mod}\,\Lambda_0$. In fact if we allow ourselves an additional tilt, then we can arrange for our hopes to be true as long as $[\underline{K}, \overline{K}]$ contains *at least one* algebraic heart, e.g. if either if the two end points are algebraic.

Lemma 2.6. *If either $\overline{K}$ or $\underline{K}$ is algebraic, i.e. fits the description (h1) in theorem 2.4, then there is an autoequivalence $\Phi : \mathbf{D}^{\mathrm{fl}}\Lambda_0 \to \mathbf{D}^{\mathrm{fl}}\Lambda_0$ composed of shifts, mutation functors, and their inverses such that x lies in $\Phi(H)[0, n-1]$.*

Before moving on to the proof we sketch an illustrative albeit contrived example. Writing $s_0, s_1 \in \mathrm{fl}\mathscr{mod}\,\Lambda_1$ for the two simple modules indexed as in [DW25, §3.1], the non–zero cohomologies of $x = \Phi_0 s_0[3] \oplus \Phi_0 s_1[1] \in \mathbf{D}^{\mathrm{fl}}\Lambda_0$ with respect to H are given by $H^0(x) = \Phi_0 s_1[1]$ and $H^{-3}(x) = \Phi_0 s_0$. In particular $H^0(x)$ is simple in H and $H^{-3}(x)$ generates the torsion–free class $H^0(x)^{\perp}$, so the only heart in $[H, H[1]]$ which improves the spread of $x \in H[\![0, 3]\!]$ is $\underline{K} = \overline{K} = \Phi_0(\mathrm{fl}\mathscr{mod}\,\Lambda_1)[1]$. But looking at $x' = \Phi_0^{-1}(x)[-1] \in H'[\![0, 2]\!]$, where $H' = \mathrm{fl}\mathscr{mod}\,\Lambda_1$, it is easy to see that further tilting H' in any torsion class containing s_1 will not worsen the spread of x'. Thus x' has spread $\leqslant 2$ with respect to each of the hearts $\Phi_0(\mathrm{fl}\mathscr{mod}\,\Lambda_0)[1]$, $\Phi_0\Phi_{-1}(\mathrm{fl}\mathscr{mod}\,\Lambda_{-1})$, $\Phi_0\Phi_{-1}\Phi_{-2}(\mathrm{fl}\mathscr{mod}\,\Lambda_{-2})$, ... It follows that each of the objects $(\Phi_0\Phi_{-1}\ldots\Phi_{N-1})^{-k} \circ \Phi_0^{-1}\Phi_0^{-1}x[-2]$ $(k \geqslant 0)$ lies in $H[0, 2]$.

Proof of lemma 2.6. The hypothesis guarantees that there is a heart $\Psi(\mathrm{fl}\mathscr{mod}\,\Lambda_i) \in [H, H[1]]$, where Ψ is composed of shifts and (inverse) mutation functors, such that x lies in $\Psi(\mathrm{fl}\mathscr{mod}\,\Lambda_i)[\![0, n-1]\!]$. Equivalently, $x' = \Psi^{-1}(x)$ lies in $H'[\![0, n-1]\!]$ where $H' = \mathrm{fl}\mathscr{mod}\,\Lambda_i$.

Now the same argument as in lemma 2.2 shows that the poset of tilted hearts $[H', H'[1]]$ contains intervals

$$[H', \overline{K}'] = \{K \in [H', H'[1]] \mid x \in K[\![0, n-1]\!]\}, \qquad [\underline{K}', H'[1]] = \{K \in [H', H'[1]] \mid x \in K[\![-1, n-2]\!]\}$$

defined using the top and bottom cohomologies of x' with respect to H'. The vanishing of $\mathrm{Hom}^{<0}(x', x')$ implies the inequality $\underline{K}' \leqslant \overline{K}'$ as in lemma 2.3.

This implies that the union $[H', \overline{K}'] \cup [\underline{K}', H'[1]]$ contains an increasing chain from H', with each K in the chain satisfying $x' \in K[m, m+n-1]$ for some $m \in \{0, -1\}$. Since an algebraic heart cannot cover or be covered by a non-algebraic heart [Shi25, corollary 4.7] and the Hasse quiver of algebraic hearts in $[H', H'[1]]$ is identical to (!) with indices shifted up by i, we see that the chain must contain some K of the form $\Phi(\mathrm{fl}\mathscr{mod}\,\Lambda_0)$ where Φ is composed of mutation functors. Then $\Phi^{-1}(x')[-m] = \Phi^{-1}\Psi^{-1}(x)[-m]$ lies in $(\mathrm{fl}\mathscr{mod}\,\Lambda_0)[0, n-1]$ as required. □

§ 2.3 Proverbial spanners in the inductive works. We now address complexes for which the induction outlined in § 2.2 fails to continue, i.e. complexes $x \in H[\![0, n]\!]$ $(n \geqslant 2)$ for which both $\underline{K}$ and $\overline{K}$ are non-algebraic and fit the descriptions (h2) or (h3) in theorem 2.4.

Note that hearts described in (h2) are clearly incomparable (under the coaisle–containment order) with those in (h3), so replacing (x, Λ_0, X) with $(\Phi_{-1}^{-1}x, \Lambda_{-1}, W)$ if necessary we may assume the hearts $\underline{K}, \overline{K}$ are both (without loss of generality) tilts of $\mathscr{coh}\,X$ in some torsion classes containing only sheaves with zero–dimensional support.

We show in this case that x fits the description (c2) in theorem A, i.e. decomposes into shifts of sheaves contained in the extension–closure $\langle \mathscr{O}_p \mid p \in C \rangle$. It is classical [see e.g. Stacks, tag 00KY] that such sheaves (henceforth dubbed *zero–dimensional sheaves*) can be characterised by the property that their support is zero–dimensional, equivalently the support (being closed in an irreducible curve) has non-empty complement in C. Support can detected by looking at morphisms from skyscrapers (and extensions thereof) as follows.

Lemma 2.7. *Suppose non-zero sheaves $u, y \in \mathscr{coh}\,X$ (supported within C) are such that $u \in \langle \mathscr{O}_p \mid p \in X \rangle$, and either $\mathrm{Hom}(y, u) = 0$ or $\mathrm{Hom}(u, y) = \mathrm{Ext}^1(u, y) = 0$ holds. Then $\dim(\mathrm{Supp}\,y) = 0$.*

Proof. The conclusion is immediate from the first hypothesis since if y was supported everywhere on C then for any $p \in \mathrm{Supp}(u)$ we would have a non-zero morphism $y \twoheadrightarrow \mathscr{O}_p \hookrightarrow u$.

To deduce this from the second hypothesis, suppose for the sake of contradiction that $\dim(\mathrm{Supp}\,y) = 1$ and $\mathrm{Hom}(u, y) = \mathrm{Ext}^1(u, y) = 0$. Thus in particular $\mathscr{Hom}(u, y) = 0$ where we note that the homomorphism sheaf is supported on a finite set of closed points $\mathrm{Supp}(u)$ so vanishes if and only if its module of global sections does.

Likewise we can deduce (e.g. from the Grothendieck spectral sequence associated to $H^0(X, -) \circ \mathbf{R}\mathscr{Hom}(u, -)$) that $H^0(X, \mathscr{Ext}^1(u, y)) = \mathrm{Ext}^1(u, y) = 0$, and hence $\mathscr{Ext}^1(u, y) = 0$.

Taking stalks at $p \in \mathrm{Supp}(u)$ we thus compute the equality of $\mathcal{O}_{X,p}$–modules $\mathrm{Hom}(u_p, y_p) = \mathrm{Ext}^1(u_p, y_p) = 0$, here the stalks u_p, y_p are finitely generated modules over the local ring $\mathcal{O}_{X,p}$ and u_p is supported on the maximal ideal (in fact is filtered by the residue field.) It follows that $\mathrm{depth}(y_p) \geqslant 2$ [Mat08, theorem 16.6] but this is absurd since the depth of a module is bounded above by the dimension of its support [Stacks, tag 00LK]. □

The result above can be extended to objects of the heart $H = \mathrm{VdB}^{-1}(\mathrm{fl}\,\mathit{mod}\,\Lambda_0)$.

Lemma 2.8. *An object* $y \in H$ *lies in the extension closure* $\langle \mathcal{O}_p \mid p \in X\rangle$ *if any of the following conditions hold.*

(i) $\mathrm{Hom}(u, y) = \mathrm{Ext}^1(u, y) = 0$ *for some non-zero* $u \in \langle \mathcal{O}_p \mid p \in C\rangle$.

(ii) $\mathrm{Hom}(y, u) = \mathrm{Ext}^1(y, u) = 0$ *for some non-zero* $u \in \langle \mathcal{O}_p \mid p \in C\rangle$.

(iii) $\mathrm{Hom}(u, y) = \mathrm{Hom}(y, v) = 0$ *for some (not necessarily distinct) non-zero* $u, v \in \langle \mathcal{O}_p \mid p \in C\rangle$.

Proof. Recall [see e.g. VdB04, §3.1] that H (when considered a subcategory of $\mathbf{D}^0X \simeq \mathbf{D}^{\mathrm{fl}}\Lambda_0$) coincides with the tilt ${}^0\mathrm{Per}(X/Z) \cap \mathbf{D}^0X$ of $\mathit{coh}\,X$, hence the object $y \in H$ sits in an exact triangle

$$y'[1] \to y \to y'' \to y'[2]$$

where $y', y'' \in \mathit{coh}\,X$ satisfy $\pi_*(y') = \mathbf{R}^1\pi_*(y'') = 0$ and $\mathrm{Hom}(\mathcal{O}_C(-1), y') = 0$. In particular y' has no subsheaves with zero-dimensional support, i.e. $\mathrm{Hom}(u, y') = 0$ for all $u \in \langle \mathcal{O}_p \mid p \in C\rangle$.

For a non-zero $u \in \langle \mathcal{O}_p \mid p \in C\rangle$, a straightforward long exact sequence argument shows there is an inclusion $\mathrm{Ext}^1(u, y') \subseteq \mathrm{Hom}(u, y)$, thus if either (i) or (iii) holds then the hypothesis $\mathrm{Hom}(u, y) = 0$ implies $\mathrm{Ext}^1(u, y') = 0$ which cannot happen unless $\dim(\mathrm{Supp}\, y') < 1$ by lemma 2.7. This necessarily implies $y' = 0$, and hence $y = y''$ is a sheaf. The same lemma then lets us deduce the condition on $\mathrm{Supp}(y)$ from either pair of hypotheses.

On the other hand if (ii) holds, then the vanishing of $\mathrm{Hom}(y'', u) = \mathrm{Hom}(y, u)$ shows $\mathrm{Supp}(y'')$ is disjoint from $\mathrm{Supp}(u)$. The long exact sequence $\ldots \to \mathrm{Ext}^1(y, u) \to \mathrm{Hom}(y', u) \to \mathrm{Ext}^2(y'', u) \to \ldots$ then shows $\mathrm{Hom}(y', u)$ vanishes too, which can only happen if $y' = 0$ and hence $y = y''$ is of the claimed form. □

We are now sufficiently equipped to witness the decomposition of x — the constraints on $\overline{K}$ show the cohomology object $H^0(x) \in H$ surjects onto a skyscraper sheaf, which we then exploit against lemma 2.8.

Lemma 2.9. *Suppose* $x \in H[\![0, n]\!]$ *(*$n \geqslant 2$*) is such that the hearts* $\underline{K}, \overline{K}$ *are both tilts of* $\mathit{coh}\,X$ *in (possibly trivial) torsion classes containing sheaves with zero–dimensional support. Then each cohomology object* $H^i(x) \in H$ *lies in* $\langle \mathcal{O}_p \mid p \in X\rangle$, *and cohomologies in distinct degrees have disjoint supports.*

Proof that $H^0(x)$ *and* $H^{-n}(x)$ *have zero–dimensional supports.* The conditions on $\overline{K}$ guarantee the containments of torsion classes

$$\mathit{coh}\,X[1] \cap H \quad \subsetneq \quad \overline{K} \cap H \quad \subseteq \quad \langle \mathit{coh}\,X[1] \cap H, \{\mathcal{O}_p \mid p \in C\}\rangle,$$

and the first containment must be proper — indeed, by [Shi25, lemma 5.16] the torsion class $\mathit{coh}\,X[1] \cap H$ is the supremum (i.e. union) of a strictly increasing chain $0 \subset T_1 \subset T_2 \subset \ldots$ in $\mathrm{tors}(H)$, so if $\overline{K} \cap H$ is equal to this union then $H^0(x) \in \overline{K} \cap H$ is contained in some T_i. But this is absurd, since $\overline{K} \cap H$ is by definition the minimal torsion class containing $H^0(x)$.

Thus $H^0(x)$ lies in $\langle \mathit{coh}\,X[1] \cap H, \{\mathcal{O}_p \mid p \in C\}\rangle \setminus (\mathit{coh}\,X[1] \cap H)$, sitting in a triangle $y[1] \to H^0(x) \to z \to y[2]$ where $y, z \in \mathit{coh}\,X$ are sheaves such that $y \in H[-1]$, and $z \in \langle \mathcal{O}_p \mid p \in C\rangle$ is non–zero.

Now the analogous condition $\underline{K}[-1] \cap H \subseteq \mathit{coh}\,X \cap H$ implies that $H^{-n}(x)$ is a sheaf, so $\mathrm{Hom}^{<0}(y, H^{-n}x)$ vanishes and the above exact triangle gives the inclusions $\mathrm{Hom}(z, H^{-n}x[i]) \subseteq \mathrm{Hom}(H^0x, H^{-n}x[i])$ for $i = 0, 1$ via a long exact sequence.

But the conditions $\mathrm{Hom}(x, x[-n]) = \mathrm{Hom}(x, x[1-n]) = 0$, after truncating to the aisle and the coaisle of H as in the proof of lemma 2.3, imply

$$\text{(!!)} \qquad \mathrm{Hom}(H^0x, H^{-n}x) = \mathrm{Hom}^1(H^0x, H^{-n}x) = 0.$$

Thus we conclude $\mathrm{Hom}(z, H^{-n}x) = \mathrm{Hom}^1(z, H^{-n}x) = 0$, so that $H^{-n}(x)$ is a sheaf with zero–dimensional support outside $\mathrm{Supp}(z)$ by lemma 2.8. It immediately follows (from (!!) and lemma 2.8) that $H^0(x)$ is also a sheaf with zero–dimensional support, in particular $H^0(x) = z$ and $y = 0$. □

Proof that the remaining cohomologies have zero–dimensional supports. One can continue to use truncation functors and analyse long exact sequences to compare the cohomologies $H^i(x)$ against $H^0(x)$ and $H^{-n}(x)$, we find it slightly more convenient to instead examine the spectral sequence [see e.g. GM96, IV.2, Exercise 2]

$$E_2^{p,q} = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^p(H^i x, H^{i+q}x) \Longrightarrow \mathrm{Hom}^{p+q}(x, x)$$

and inductively show enough rows vanish on the second page. Suppose we have shown that the terms $E_2^{p,q}$ all vanish for $q \leqslant j$, from the above discussion this is true for $j = -n$. Since the terms also vanish if $p \leqslant 0$, we see that the differentials in and out of $E_2^{0,1+j}$ and $E_2^{1,1+j}$ are trivial so that these terms persist in future pages of the sequence, arising as subquotients of $\mathrm{Hom}^{1+j}(x, x)$ and $\mathrm{Hom}^{2+j}(x, x)$ respectively.

In particular if $j < -2$, then $E_2^{0,1+j} = E_2^{1,1+j} = 0$ and we have

$$\begin{aligned} \mathrm{Hom}\ (H^0x, H^{1+j}x) &= \mathrm{Hom}\ (H^{-n-j-1}x, H^{-n}x) = 0, \\ \mathrm{Hom}^1(H^0x, H^{1+j}x) &= \mathrm{Hom}^1(H^{-n-j-1}x, H^{-n}x) = 0. \end{aligned}$$

Since $H^0(x)$ and $H^{-n}(x)$ both lie in $\langle \mathcal{O}_p \mid p \in X \rangle$, lemma 2.8 and the above equalities show that the same holds for $H^{1+j}(x)$ and $H^{-n-j-1}(x)$. In fact an identical reasoning for lower rows shows $H^i(x) \in \langle \mathcal{O}_p \mid p \in X \rangle$ for each of the values $i \in \{-n, -n+1, ..., j+1\} \cup \{-n-j-1, -n-j, ..., 0\}$. For these values of i, the vanishing of $\mathrm{Hom}(H^i x, H^{i+j+1}x) \subset E_2^{0,1+j}$ then shows that $H^i(x)$ and $H^{i+j+1}(x)$ have disjoint supports, so that $E_2^{p,j+1} = 0$ for remaining values of p too.

Thus for each $q \leqslant -2$, we have $E_2^{0,q} = E_2^{1,q} = 0$, and hence $H^q(x)$ and $H^{-n-q}(x)$ have zero–dimensional supports as above.

If $n \geqslant 3$ this accounts for all cohomologies of x. If $n = 2$ the above argument fails to address the cohomology in degree -1, but note that in this case the vanishing of $E_2^{0,-1} = \mathrm{Hom}(H^0x, H^{-1}x) \oplus \mathrm{Hom}(H^{-1}x, H^{-2}x)$ alone suffices to apply lemma 2.8 and deduce that $\dim(\mathrm{Supp}(H^{-1}x)) = 0$. □

Proof that cohomologies have mutually disjoint supports. Since x is an extension of shifts of skyscrapers, it admits a decomposition $x = \bigoplus_{p \in C} x|_p$ indexed over closed points of C, where the summand $x|_p$ is a complex supported within $\{p\}$. In particular the top non–zero cohomology of $x|_p$ admits a non-trivial morphism (which factors via $\mathcal{O}_p$) to the bottom non–zero cohomology of $x|_p$. But $x|_p$ also satisfies $\mathrm{Hom}^{<0}(x|_p, x|_p) = 0$, whence $x|_p$ must be a sheaf concentrated in a single cohomological degree. Since $H^i(x) = \bigoplus_{p \in C} H^i(x|_p)$, the result follows. □

The above result, combined with lemma 2.6 and corollary 2.5, finishes our analysis of all complexes supported on C that have only non–negative self extensions, describing a complete proof of theorem A.

§ 3 Corollaries, corollaries

A *brick* in $\mathbf{D}^0 X$ is a complex x whose self–extensions mimic those of a simple module, i.e. $\mathrm{Hom}^{<0}(x,x) = 0$ and $\mathrm{Hom}(x,x) = \mathbb{C}$. A *semibrick* is a set $\{x_i \mid i \in I\}$ of bricks that satisfy the orthogonality $\mathrm{Hom}^{\leqslant 0}(x_i, x_j) = 0$ whenever $i \neq j$. A *simple–minded collection* is a finite semibrick that is not contained in a proper thick subcategory of $\mathbf{D}^0 X$. The heart of an algebraic t-structure is the extension–closure of a simple–minded collection, conversely every simple–minded collection is the set of simples of an algebraic heart [Al-09].

We sketch how finite semibricks, hence also bricks, simple–minded collections, and algebraic t-structures in $\mathbf{D}^0 X$, are all readily classified using theorem A. Throughout, the phrase *up to mutation* means up to shifts and iterated application of functors appearing in $(*)$ and their inverses.

Corollary 3.1. *Every finite semibrick* $\{x_0, \dots, x_m\} \subset \mathbf{D}^0 X$ *must, up to mutation, either be a collection of simple* Λ_i*−modules for some* $i \in \mathbb{Z}/N\mathbb{Z}$*, or be of the form* $\{\mathcal{O}_{p_0}[n_0], \dots, \mathcal{O}_{p_m}[n_m]\}$ *for distinct closed points* $p_0, \dots, p_m$ *in* X *(or in* W*) and integers* $n_0, \dots, n_m \in \mathbb{Z}$.

Consequently every brick in $\mathbf{D}^0 X$ *is, up to mutation, a simple* Λ_i*−module for some* $i \in \mathbb{Z}/N\mathbb{Z}$*, or a skyscraper sheaf* $\mathcal{O}_p$ *at a closed point in* X *or in* W*. Likewise, any heart of an algebraic t-structure in* $\mathbf{D}^0 X$ *is, up to mutation, the standard heart* $\mathrm{fl}\mathcal{mod}\,\Lambda_i \subset \mathbf{D}^{\mathrm{fl}} \Lambda_i$ *for some* $i \in \mathbb{Z}/N\mathbb{Z}$.

Proof. Consider the object $x = x_0 \oplus \dots \oplus x_m$, which clearly satisfies $\mathrm{Hom}^{<0}(x,x) = 0$ and is therefore classified by theorem A. If it fits the description (c2), then each x_i must be a shift of some sheaf $\mathcal{O}_p$ $(p \in X)$ since those are the only brick sheaves with zero–dimensional support (to see this, note all objects in $\langle \mathcal{O}_p \rangle$ have a non-zero endomorphism which factors via $\mathcal{O}_p$).

On the other hand if x fits the description (c1), i.e. the semibrick $\{x_0, \dots, x_m\}$ up to mutation lies in some $H' = \mathrm{fl}\mathcal{mod}\,\Lambda_j$, then consider the smallest torsion class $T \subset H'$ that contains x. By [Shi25, proposition 2.6], each x_i is a simple object in the heart K obtained by tilting H' in this torsion class. Examining the Hasse quiver of $[H', H'[1]]$ as sketched in § 2.2, we see that K must also be a tilt of $\Phi(\mathrm{fl}\mathcal{mod}\,\Lambda_0)[1]$ or $\Phi^{-1}(\mathrm{fl}\mathcal{mod}\,\Lambda_0)$ for some mutation functor Φ. Thus K is, up to mutation, a tilt of $\mathrm{fl}\mathcal{mod}\,\Lambda_0$ and required descriptions of the simple objects $x_0, \dots, x_m \in K$ can be read off from theorem 2.4.

To address the last case, suppose x is a two–term complex of coherent sheaves as in (c3), without loss of generality on X. Each summand x_i must then also admit the same description, sitting in an exact triangle

$$\mathcal{F}_i[1] \to x_i \to \mathcal{G}_i \to \mathcal{F}_i[2]$$

with $\mathcal{G}_i \in \langle \mathcal{O}_p \mid p \in X \rangle$, $\mathcal{F}_i \in \mathcal{coh}\,X$, and $\mathrm{Hom}(\mathcal{G}_i, \mathcal{F}_i) = 0$.

Claim for each i, either $\mathcal{F}_i$ or $\mathcal{G}_i$ vanishes. Indeed if $\mathcal{F}_i \neq 0$, it cannot be a summand in $\mathcal{G}_i[-2]$ so the first map in the triangle above is a non–zero element in $\mathrm{Hom}(\mathcal{F}_i[1], x_i)$. This element is annihilated by the natural map $\mathrm{Hom}(\mathcal{F}_i[1], x_i) \to \mathrm{Hom}(\mathcal{G}_i[-1], x_i)$ since any two consecutive maps in an exact triangle compose to zero [Stacks, tag 0146]. Thus in the long exact sequence

$$\dots \to \underbrace{\mathrm{Hom}(\mathcal{F}_i[2], x_i)}_{0} \longrightarrow \mathrm{Hom}(\mathcal{G}_i, x_i) \longrightarrow \underbrace{\mathrm{Hom}(x_i, x_i)}_{\mathbb{C}} \xrightarrow{\alpha} \mathrm{Hom}(\mathcal{F}_i[1], x_i) \longrightarrow \mathrm{Hom}(\mathcal{G}_i[-1], x_i) \to \dots$$

the final map has non-trivial kernel so that α is injective. It follows that $\mathrm{Hom}(\mathcal{G}_i, x_i) = 0$, and considering the long exact sequence associated to $\mathrm{Hom}(\mathcal{G}_i, -)$ shows $\mathrm{Hom}(\mathcal{G}_i, \mathcal{F}_i[1])$ vanishes too. If $\mathcal{G}_i$ were non–zero then lemma 2.8 would show $\mathrm{Supp}(\mathcal{F}_i)$ is zero–dimensional hence disjoint from $\mathrm{Supp}(\mathcal{G}_i)$, but this is absurd since we would then have a non–trivial splitting $x_i = \mathcal{F}_i[1] \oplus \mathcal{G}_i$ of a brick.

Thus there is a $k \geqslant 0$ and a reordering of the summands of x so that $x_i = \mathcal{G}_i$ for $i < k$, and $x_i = \mathcal{F}_i[1]$ for $i \geqslant k$. If $k \geqslant 1$ then lemma 2.7 shows each $\mathcal{F}_i$ $(i \geqslant k)$ has zero–dimensional support since $\mathrm{Ext}^{-1}(x_0, x_i) = \mathrm{Hom}(\mathcal{G}_0, \mathcal{F}_i)$ and $\mathrm{Hom}(x_0, x_i) = \mathrm{Ext}^1(\mathcal{G}_0, \mathcal{F}_i)$ both vanish, whence x also admits the description (c2) and can be addressed as in the first paragraph.

On the other hand if $k = 0$ (so that $x \in \mathcal{coh}\,X[1]$), then we can consider (as in lemma 2.3) the maximal tilt $\overline{K}$ of $H = \mathrm{fl}\,\mathcal{mod}\,\Lambda_0$ satisfying $x \in \overline{K}$. We have $\overline{K} \geqslant \mathcal{coh}\,X[1]$ (i.e. $\overline{K} \cap H \subseteq \mathcal{coh}\,X[1] \cap H$), and equality cannot hold since the torsion class $\mathcal{coh}\,X[1] \cap H$ is the union of an increasing chain in $\mathrm{tors}(H)$ while the torsion class $\overline{K} \cap H$ is the intersection of all torsion classes containing $H^0(x)$ (see proof of lemma 2.9). It follows from theorem 2.4 that $\overline{K}$ is an algebraic heart, so that $x \in \overline{K}$ also admits the description (c1) and is addressed by the second paragraph. □

The classification of algebraic t-structures in $\mathbf{D}^0X$ can be lifted to a classification of tilting complexes in the bigger category $\mathbf{D}^bX$, following [HW23, §5.4]. Recall that a complex $t \in \mathbf{D}^bX$ is *tilting* if it satisfies $\mathrm{Hom}^i(t,t) = 0$ for all $i \neq 0$ and the smallest thick subcategory of $\mathbf{D}^bX$ containing t also contains all perfect complexes.

Given an algebra Λ and a derived equivalence $\Psi : \mathbf{D}^b\Lambda \to \mathbf{D}^bX$, the image $\Psi(\Lambda)$ of the regular Λ–module is a tilting object in $\mathbf{D}^bX$. The converse is Rickard's theorem [Ric89, theorem 6.4], that any tilting object $t \in \mathbf{D}^bX$ arises in the above fashion from the R–algebra $\Lambda = \mathrm{End}(t)$ and the R–linear equivalence determined as $\Psi^{-1} = \mathbf{R}\mathrm{Hom}(t,-)$. If the complex t (equivalently the algebra Λ) is *basic*, i.e. if distinct indecomposable summands of t are non–isomorphic, then Morita theory[2] [see e.g. Lam99, proposition 18.37] picks out t as the unique basic projective generator in the heart $\Psi(\mathcal{mod}\,\Lambda) \subset \mathbf{D}^bX$.

In this fashion, each equivalence $\mathbf{D}^b\Lambda_i \to \mathbf{D}^bX$ composed of the functors appearing in $(*)$ and their inverses gives a unique basic tilting object in $\mathbf{D}^bX$. These tilting objects sit in a connected tetravalent exchange graph so that each one can be obtained from $\mathrm{VdB}^{-1}(\Lambda_0) = \mathcal{N} \oplus \mathcal{O}_X$ by iterated left or right mutation in indecomposable summands (i.e. by *simple mutation*). Other tilting complexes include non–trivial shifts of those already considered, these can be obtained by simple mutation from $(\mathcal{N} \oplus \mathcal{O}_X)[n]$. We show this list is exhaustive.

Corollary 3.2. *Any basic tilting complex* $t \in \mathbf{D}^bX$ *can be obtained by iteratively mutating the tilting complex* $(\mathcal{N}\oplus\mathcal{O}_X)[n]$ *in indecomposable summands, for some* $n \in \mathbb{Z}$.

Proof. Consider the module–finite R–algebra $\Lambda = \mathrm{End}(t)$ and the induced equivalence $\Psi : \mathbf{D}^b\Lambda \to \mathbf{D}^bX$ with inverse $\mathbf{R}\mathrm{Hom}(t,-)$. By R–linearity, the functor Ψ restricts to an equivalence $\Psi : \mathbf{D}^{\mathrm{fl}}\Lambda \to \mathbf{D}^0X$ and hence the image of $\mathrm{fl}\,\mathcal{mod}\,\Lambda$ is an algebraic heart in $\mathbf{D}^0X$. By corollary 3.1, we thus see that $\Psi(\mathrm{fl}\,\mathcal{mod}\,\Lambda)$ agrees with some heart $\Phi(\mathrm{fl}\,\mathcal{mod}\,\Lambda_i)[n]$ for $i \in \mathbb{Z}/N\mathbb{Z}$, $n \in \mathbb{Z}$, and Φ a composite of functors appearing in $(*)$ and their inverses.

But by [HW23, lemma 5.7], the subcategory $\mathrm{fl}\,\mathcal{mod}\,\Lambda \subset \mathbf{D}^b\Lambda$ determines the natural t-structure via

$$\mathcal{mod}\,\Lambda\,[\geqslant 0] = \left\{ x \in \mathbf{D}^b\Lambda \mid \mathrm{Hom}^j(x,y) = 0 \text{ for all } j < 0 \text{ and all } y \in \mathrm{fl}\,\mathcal{mod}\,\Lambda \right\}.$$

Likewise the subcategory $\mathrm{fl}\,\mathcal{mod}\,\Lambda_i \subset \mathbf{D}^b\Lambda_i$ determines the natural t-structure with heart $\mathcal{mod}\,\Lambda_i$, thus we see that the the equality $\Psi(\mathrm{fl}\,\mathcal{mod}\,\Lambda) = \Phi(\mathrm{fl}\,\mathcal{mod}\,\Lambda_i)[n]$ extends to an equality $\Psi(\mathcal{mod}\,\Lambda) = \Phi(\mathcal{mod}\,\Lambda_i)[n]$. In particular $t = \Psi(\Lambda)$ must equal the basic projective generator $\Phi(\Lambda_i)[n] \in \Phi(\mathcal{mod}\,\Lambda_i)[n]$, so that t is obtained by iterated simple mutation from $(\mathcal{N} \oplus \mathcal{O}_X)[n]$ as required. □

Note that the shift n appearing above is unique, and can be read off as the cohomological degree of the generic stalk $\Bbbk \otimes_R t$ where $\Bbbk = \mathbb{C}(Z)$ is the function field of Z. Indeed considering $t \in \mathbf{D}^b\Lambda_0$ (across Van den Bergh's equivalence), the localisation $\Bbbk \otimes t$ is a tilting module over $\Bbbk \otimes \Lambda_0$ [Nam23, lemma 6.4]. But $\Bbbk \otimes \Lambda_0$ is a matrix ring over the field $\Bbbk$ (see lemma 6.2 *ibid*) so every tilting $(\Bbbk \otimes \Lambda_0)$–module is the shift of a $\Bbbk$–vector space. It follows that t is obtained by simple mutation from $\Lambda_0[n]$ if and only if $\Bbbk \otimes t[-n] = \Bbbk \otimes \Lambda_0$ is a $\Bbbk$–vector space.

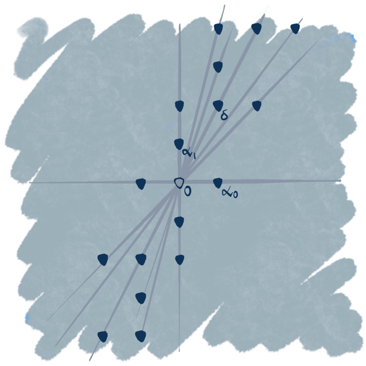
We conclude by addressing the connectivity of the stability manifold of $\mathbf{D}^0X$, referring the reader to [Bri06, §3] for definitions and generalities. The stability manifold $\mathrm{Stab}(\mathbf{D}^0X)$ has been extensively analysed [Bri09; Tod08; HW23] with a key result being the existence of a connected component $\mathrm{Stab}^\circ(\mathbf{D}^0X)$ that contains all stability conditions (Z, P) for which some heart $P(\varphi, \varphi + 1]$ agrees with some $\mathrm{fl}\,\mathcal{mod}\,\Lambda_j$ up to mutation, equivalently is algebraic (corollary 3.1). We show that every locally finite stability condition on $\mathbf{D}^0X$ is of this form.

[2] This requires Λ to be semi–perfect, which is true since it is a module–finite algebra over the complete local domain R.

Corollary 3.3. *For any locally finite Bridgeland stability condition* (Z, P) *on* $\mathbf{D}^0X$*, there is a phase* $\varphi \in \mathbb{R}$ *such that the heart* $P(\varphi, \varphi + 1]$ *is algebraic. In particular the stability manifold* $\operatorname{Stab}(\mathbf{D}^0X) = \operatorname{Stab}^\circ(\mathbf{D}^0X)$ *is connected.*

Proof. Given the stability condition $\sigma = (Z, P)$, consider the set $\{\varphi \in \mathbb{R} \mid P(\varphi) \neq \emptyset\}$ of phases occupied by σ–stable objects. Now a σ–stable object $x \in \mathbf{D}^0X$ of phase φ is necessarily simple in the Abelian category $P(\varphi)$ and therefore is a brick, equal to a simple Λ_i–module or a skyscraper sheaf up to mutation (corollary 3.1).

But [Shi25, lemma 4.18] then shows that the **K**–theory class $[x]$ is a restricted affine root in the rank two lattice $\mathbf{K}(\mathbf{D}^0X)$, that is to say there is a root system $\Pi \subset \mathbb{Z}^{\oplus m}$ associated to a Cartan matrix of affine type, and a projection $\mathbb{Z}^{\oplus m} \twoheadrightarrow \mathbb{Z}^{\oplus 2} \simeq \mathbf{K}(\mathbf{D}^0X)$ which maps a root $\alpha \in \Pi$ to $[x]$. Such restricted root systems have been studied in [IW, chapter 2], they define affine hyperplane arrangements in $\mathbb{R}^2$ that are locally finite away from the origin. It follows that the possible phases of the complex number $Z[x]$ must be restricted to a countable set of values with a discrete set of accummulation points. The example alongside is the affine D_4 restricted root system associated to a length 2 flop — clearly all phases accummulate towards that of $\pm\delta$.



In particular σ has a 'non-trivial phase gap', i.e. there is a $\varphi \in \mathbb{R}$ and $\epsilon > 0$ such that $P(\varphi, \varphi + \epsilon) = \emptyset$. If the image of $Z : \mathbf{K}(\mathbf{D}^0X) \to \mathbb{C}$ is a discrete subgroup of $\mathbb{C}$, then we may use [Bri08, lemma 4.4] to conclude that $P(\varphi, \varphi + 1] = P(\varphi + \epsilon, \varphi + 1]$ is an algebraic Abelian category. On the other hand if $\operatorname{img}(Z)$ is a non–discrete rank 2 subgroup of $\mathbb{C}$, then this image must lie in the $\mathbb{R}$–span of some non–zero complex number $e^{i\pi\varphi}$ so that the phase of each stable object is an integer away from φ. Then local finiteness allows us to conclude that $P(\varphi, \varphi + 1] = P(\varphi + 1)$ is algebraic. □

References

[Al-09] Salah Al-Nofayee. "Simple Objects in the Heart of a T-Structure". In: *Journal of Pure and Applied Algebra* 213.1 (2009). DOI: 10.1016/j.jpaa.2008.05.014.

[AM17] Takuma Aihara and Yuya Mizuno. "Classifying Tilting Complexes over Preprojective Algebras of Dynkin Type". In: *Algebra & Number Theory* 11.6 (2017). arXiv: 1509.07387.

[Asp03] Paul S. Aspinwall. "A Point's Point of View of Stringy Geometry". In: *Journal of High Energy Physics* 2003.01 (2003). arXiv: hep-th/0203111.

[Aug20] Jenny August. "On the Finiteness of the Derived Equivalence Classes of Some Stable Endomorphism Rings". In: *Mathematische Zeitschrift* 296.3-4 (2020). arXiv: 1801.05687.

[Bri02] Tom Bridgeland. "Flops and Derived Categories". In: *Inventiones Mathematicae* 147.3 (2002). arXiv: math/0009053.

[Bri06] Tom Bridgeland. *Spaces of Stability Conditions.* 2006. arXiv: math/0611510.

[Bri08] Tom Bridgeland. "Stability Conditions on K3 Surfaces". In: *Duke Mathematical Journal* 141.2 (2008). arXiv: math/0307164.

[Bri09] Tom Bridgeland. "Stability Conditions and Kleinian Singularities". In: *International Mathematics Research Notices* (2009). arXiv: math/0508257.

[DW25] Will Donovan and Michael Wemyss. "Stringy Kähler Moduli, Mutation and Monodromy". In: *Journal of Differential Geometry* 129.1 (2025). arXiv: 1907.10891.

[GM96] S. I. Gel'fand and Y Manin. *Methods of Homological Algebra.* Springer, 1996.

[HRS96] Dieter Happel, Idun Reiten, and Sverre O. Smalø. *Tilting in Abelian Categories and Quasitilted Algebras.* Memoirs of the American Mathematical Society no. 575. American Mathematical Soc., 1996.

[HW23] Yuki Hirano and Michael Wemyss. "Stability Conditions for 3-Fold Flops". In: *Duke Mathematical Journal* 172.16 (2023). arXiv: 1907.09742.

[HW24] Wahei Hara and Michael Wemyss. "Spherical Objects in Dimensions Two and Three". In: *Journal of the European Mathematical Society* (2024). arXiv: 2205.11552.

[IU05] Akira Ishii and Hokuto Uehara. “Autoequivalences of Derived Categories on the Minimal Resolutions of An-singularities on Surfaces”. In: *J. of Differential Geometry* 71.3 (2005). arXiv: math/0409151.

[IW] Osamu Iyama and Michael Wemyss. *Tits Cone Intersections and Applications.* (In preparation). URL: https://www.maths.gla.ac.uk/~mwemyss/MainFile_for_web.pdf.

[KS24] Ailsa Keating and Ivan Smith. “Symplectomorphisms and Spherical Objects in the Conifold Smoothing”. In: *Compositio Mathematica* 160.11 (2024). arXiv: 2301.10525.

[KV00] M. Kapranov and E. Vasserot. “Kleinian Singularities, Derived Categories and Hall Algebras”. In: *Mathematische Annalen* 316.3 (2000). arXiv: math/9812016.

[Lam99] T. Y. Lam. *Lectures on Modules and Rings.* Graduate Texts in Mathematics 189. Springer, 1999.

[Mat08] Hideyuki Matsumura. *Commutative Ring Theory.* Cambridge Studies in Advanced Mathematics 8. Cambridge: Cambridge Univ. Press, 2008. ISBN: 978-0-521-36764-6.

[Nam23] Caroline Namanya. *Group Actions from Algebraic Flops.* 2023. arXiv: 2310.18062.

[Ric89] Jeremy Rickard. “Morita Theory for Derived Categories”. In: *Journal of the London Mathematical Society* s2-39.3 (1989).

[RW+] Stefan Roberts and Jon Woolf. *The topology of 2-dimensional stability spaces.* (In preparation).

[Shi25] Parth Shimpi. *Torsion Pairs and 3-Fold Flops.* 2025. arXiv: 2502.05146.

[ST01] Paul Seidel and Richard Thomas. “Braid Group Actions on Derived Categories of Coherent Sheaves”. In: *Duke Mathematical Journal* 108.1 (2001). arXiv: math/0001043.

[Stacks] The Stacks project authors. *The Stacks project.* 2025. URL: https://stacks.math.columbia.edu.

[Tod08] Yukinobu Toda. “Stability Conditions and Crepant Small Resolutions”. In: *Transactions of the Americal Mathematical Society* 360.11 (2008). arXiv: math/0512648.

[VdB04] Michel Van den Bergh. “Three-Dimensional Flops and Noncommutative Rings”. In: *Duke Mathematical Journal* 122.3 (2004). arXiv: math/0207170.

[Wem18] Michael Wemyss. “Flops and Clusters in the Homological Minimal Model Programme”. In: *Inventiones Mathematicae* 211.2 (2018). arXiv: 1411.7189.

THE MATHEMATICS AND STATISTICS BUILDING, UNIVERSITY OF GLASGOW, UNIVERSITY PLACE, GLASGOW G12 8QQ, UK.

Email address: parth.shimpi@glasgow.ac.uk

Web: https://pas201.user.srcf.net